14

The Basic Elements and Phasors

14.1 INTRODUCTION

The response of the basic R, L, and C elements to a sinusoidal voltage and current will be examined in this chapter, with special note of how frequency will affect the "opposing" characteristic of each element. Phasor notation will then be introduced to establish a method of analysis that permits a direct correspondence with a number of the methods, theorems, and concepts introduced in the dc chapters.

14.2 THE DERIVATIVE

In order to understand the response of the basic R, L, and C elements to a sinusoidal signal, you need to examine the concept of the **derivative** in some detail. It will not be necessary that you become proficient in the mathematical technique, but simply that you understand the impact of a relationship defined by a derivative.

Recall from Section 10.11 that the derivative dx/dt is defined as the rate of change of x with respect to time. If x fails to change at a particular instant, $dx = 0$, and the derivative is zero. For the sinusoidal waveform, dx/dt is zero only at the positive and negative peaks ($\omega t = \pi/2$ and $\frac{3}{2}\pi$ in Fig. 14.1), since x fails to change at these instants of time. The derivative dx/dt is actually the slope of the graph at any instant of time.

A close examination of the sinusoidal waveform will also indicate that the greatest change in x will occur at the instants $\omega t = 0$, π, and 2π. The derivative is therefore a maximum at these points. At 0 and 2π, x increases at its greatest rate, and the derivative is given a positive sign since x increases with time. At π, dx/dt decreases at the same rate as it increases at 0 and 2π, but the derivative is given a negative sign since x decreases with time. Since the rate of change at 0, π, and 2π is the same, the magnitude of the derivative at these points is the same also. For various values of ωt between these maxima and minima, the derivative will exist and will have values from the minimum to the maximum inclusive. A plot of the derivative in Fig. 14.2 shows that

the derivative of a sine wave is a cosine wave.

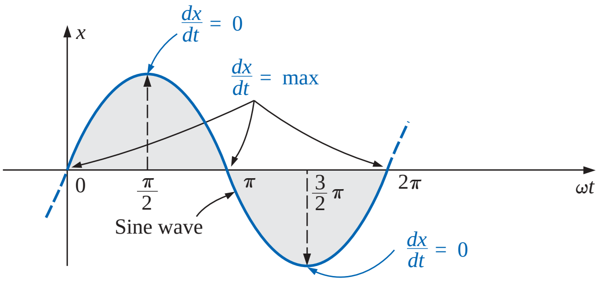

FIG. 14.1

Defining those points in a sinusoidal waveform that have maximum and minimum derivatives.

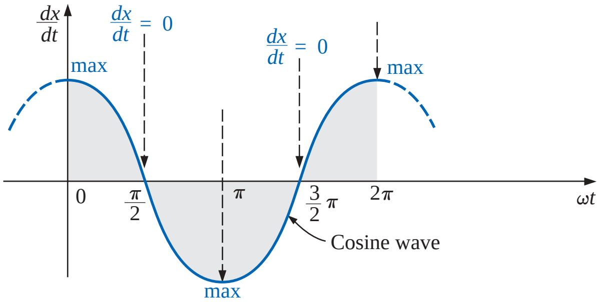

FIG. 14.2

Derivative of the sine wave of Fig. 14.1.

The peak value of the cosine wave is directly related to the frequency of the original waveform. The higher the frequency, the steeper the slope at the horizontal axis and the greater the value of dx/dt, as shown in Fig. 14.3 for two different frequencies.

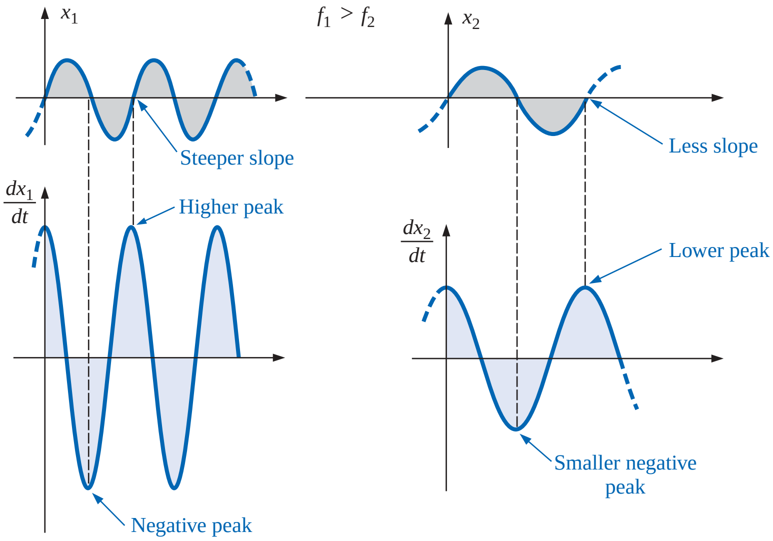

FIG. 14.3

Effect of frequency on the peak value of the derivative.

Note in Fig. 14.3 that even though both waveforms (x_1 and x_2) have the same peak value, the sinusoidal function with the higher frequency produces the larger peak value for the derivative. In addition, note that

the derivative of a sine wave has the same period and frequency as the original sinusoidal waveform.

For the sinusoidal voltage

$$e(t) = E_m \sin(\omega t \pm \theta)$$

the derivative can be found directly by differentiation (calculus) to produce the following:

$$\boxed{\begin{aligned}\frac{d}{dt}e(t) &= \omega E_m \cos(\omega t \pm \theta)\\ &= 2\pi f E_m \cos(\omega t \pm \theta)\end{aligned}} \qquad \textbf{(14.1)}$$

The mechanics of the differentiation process will not be discussed or investigated here; nor will they be required to continue with the text. Note, however, that the peak value of the derivative, $2\pi f E_m$, is a function of the frequency of $e(t)$, and the derivative of a sine wave is a cosine wave.

14.3 RESPONSE OF BASIC *R, L,* AND *C* ELEMENTS TO A SINUSOIDAL VOLTAGE OR CURRENT

Now that we are familiar with the characteristics of the derivative of a sinusoidal function, we can investigate the response of the basic elements *R*, *L*, and *C* to a sinusoidal voltage or current.

Resistor

For power-line frequencies and frequencies up to a few hundred kilohertz, resistance is, for all practical purposes, unaffected by the frequency of the applied sinusoidal voltage or current. For this frequency region, the resistor *R* of Fig. 14.4 can be treated as a constant, and Ohm's law can be applied as follows. For $v = V_m \sin \omega t$,

$$i = \frac{v}{R} = \frac{V_m \sin \omega t}{R} = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t$$

where
$$\boxed{I_m = \frac{V_m}{R}} \qquad \textbf{(14.2)}$$

In addition, for a given *i*,

$$v = iR = (I_m \sin \omega t)R = I_m R \sin \omega t = V_m \sin \omega t$$

where
$$\boxed{V_m = I_m R} \qquad \textbf{(14.3)}$$

A plot of *v* and *i* in Fig. 14.5 reveals that

for a purely resistive element, the voltage across and the current through the element are in phase, with their peak values related by Ohm's law.

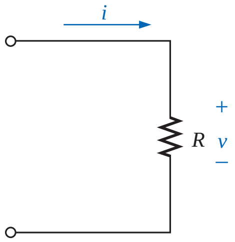

FIG. 14.4
Determining the sinusoidal response for a resistive element.

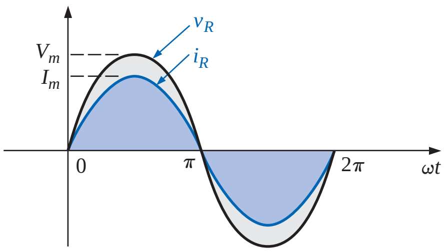

FIG. 14.5
The voltage and current of a resistive element are in phase.

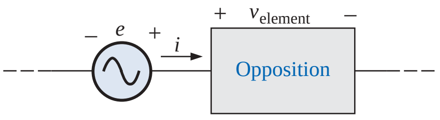

FIG. 14.6
Defining the opposition of an element to the flow of charge through the element.

Inductor

For the series configuration of Fig. 14.6, the voltage v_{element} of the boxed-in element opposes the source e and thereby reduces the magnitude of the current i. The magnitude of the voltage across the element is determined by the opposition of the element to the flow of charge, or current i. For a resistive element, we have found that the opposition is its resistance and that v_{element} and i are determined by $v_{\text{element}} = iR$.

We found in Chapter 12 that the voltage across an inductor is directly related to the rate of change of current through the coil. Consequently, the higher the frequency, the greater will be the rate of change of current through the coil, and the greater the magnitude of the voltage. In addition, we found in the same chapter that the inductance of a coil will determine the rate of change of the flux linking a coil for a particular change in current through the coil. The higher the inductance, the greater the rate of change of the flux linkages, and the greater the resulting voltage across the coil.

The inductive voltage, therefore, is directly related to the frequency (or, more specifically, the angular velocity of the sinusoidal ac current through the coil) and the inductance of the coil. For increasing values of f and L in Fig. 14.7, the magnitude of v_L will increase as described above.

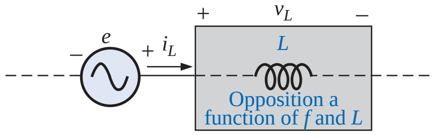

FIG. 14.7
Defining the parameters that determine the opposition of an inductive element to the flow of charge.

Utilizing the similarities between Figs. 14.6 and 14.7, we find that increasing levels of v_L are directly related to increasing levels of opposition in Fig. 14.6. Since v_L will increase with both ω ($= 2\pi f$) and L, the opposition of an inductive element is as defined in Fig. 14.7.

We will now verify some of the preceding conclusions using a more mathematical approach and then define a few important quantities to be employed in the sections and chapters to follow.

For the inductor of Fig. 14.8, we recall from Chapter 12 that

$$v_L = L\frac{di_L}{dt}$$

and, applying differentiation,

$$\frac{di_L}{dt} = \frac{d}{dt}(I_m \sin \omega t) = \omega I_m \cos \omega t$$

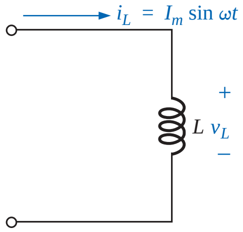

FIG. 14.8
Investigating the sinusoidal response of an inductive element.

Therefore, $$v_L = L\frac{di_L}{dt} = L(\omega I_m \cos \omega t) = \omega L I_m \cos \omega t$$

or $$v_L = V_m \sin(\omega t + 90°)$$

where $$V_m = \omega L I_m$$

Note that the peak value of v_L is directly related to ω ($= 2\pi f$) and L as predicted in the discussion above.

A plot of v_L and i_L in Fig. 14.9 reveals that

for an inductor, v_L leads i_L by 90°, or i_L lags v_L by 90°.

If a phase angle is included in the sinusoidal expression for i_L, such as

$$i_L = I_m \sin(\omega t \pm \theta)$$

then $$v_L = \omega L I_m \sin(\omega t \pm \theta + 90°)$$

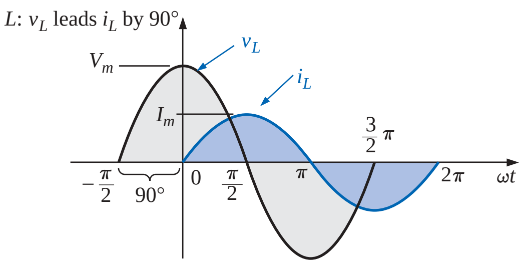

FIG. 14.9

For a pure inductor, the voltage across the coil leads the current through the coil by 90°.

The opposition established by an inductor in a sinusoidal ac network can now be found by applying Eq. (4.1):

$$\text{Effect} = \frac{\text{cause}}{\text{opposition}}$$

which, for our purposes, can be written

$$\text{Opposition} = \frac{\text{cause}}{\text{effect}}$$

Substituting values, we have

$$\text{Opposition} = \frac{V_m}{I_m} = \frac{\omega L I_m}{I_m} = \omega L$$

revealing that the opposition established by an inductor in an ac sinusoidal network is directly related to the product of the angular velocity ($\omega = 2\pi f$) and the inductance, verifying our earlier conclusions.

The quantity ωL, called the **reactance** (from the word *reaction*) of an inductor, is symbolically represented by X_L and is measured in ohms; that is,

$$\boxed{X_L = \omega L} \qquad (\text{ohms}, \Omega) \qquad \textbf{(14.4)}$$

In an Ohm's law format, its magnitude can be determined from

$$\boxed{X_L = \frac{V_m}{I_m}} \qquad (\text{ohms}, \Omega) \qquad \textbf{(14.5)}$$

Inductive reactance is the opposition to the flow of current, which results in the continual interchange of energy between the source and the magnetic field of the inductor. In other words, inductive reactance, unlike resistance (which dissipates energy in the form of heat), does not dissipate electrical energy (ignoring the effects of the internal resistance of the inductor).

Capacitor

Let us now return to the series configuration of Fig. 14.6 and insert the capacitor as the element of interest. For the capacitor, however, we will determine *i* for a particular voltage across the element. When this approach reaches its conclusion, the relationship between the voltage and current will be known, and the opposing voltage (v_{element}) can be determined for any sinusoidal current i.

Our investigation of the inductor revealed that the inductive voltage across a coil opposes the instantaneous change in current through the coil. For capacitive networks, the voltage across the capacitor is limited by the rate at which charge can be deposited on, or released by, the plates of the capacitor during the charging and discharging phases, respectively. In other words, an instantaneous change in voltage across a capacitor is opposed by the fact that there is an element of time required to deposit charge on (or release charge from) the plates of a capacitor, and $V = Q/C$.

Since capacitance is a measure of the rate at which a capacitor will store charge on its plates,

for a particular change in voltage across the capacitor, the greater the value of capacitance, the greater will be the resulting capacitive current.

In addition, the fundamental equation relating the voltage across a capacitor to the current of a capacitor [$i = C(dv/dt)$] indicates that

for a particular capacitance, the greater the rate of change of voltage across the capacitor, the greater the capacitive current.

Certainly, an increase in frequency corresponds to an increase in the rate of change of voltage across the capacitor and to an increase in the current of the capacitor.

The current of a capacitor is therefore directly related to the frequency (or, again more specifically, the angular velocity) and the capacitance of the capacitor. An increase in either quantity will result in an increase in the current of the capacitor. For the basic configuration of Fig. 14.10, however, we are interested in determining the opposition of the capacitor as related to the resistance of a resistor and ωL for the inductor. Since an increase in current corresponds to a decrease in opposition, and i_C is proportional to ω and C, the opposition of a capacitor is inversely related to ω ($= 2\pi f$) and C.

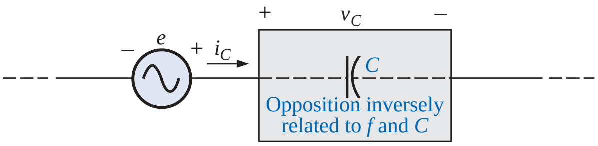

FIG. 14.10

Defining the parameters that determine the opposition of a capacitive element to the flow of the charge.

We will now verify, as we did for the inductor, some of the above conclusions using a more mathematical approach.

For the capacitor of Fig. 14.11, we recall from Chapter 10 that

$$i_C = C\frac{dv_C}{dt}$$

and, applying differentiation,

$$\frac{dv_C}{dt} = \frac{d}{dt}(V_m \sin \omega t) = \omega V_m \cos \omega t$$

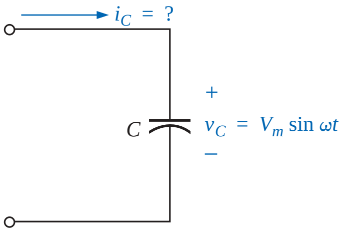

FIG. 14.11

Investigating the sinusoidal response of a capacitive element.

Therefore,

$$i_C = C\frac{dv_C}{dt} = C(\omega V_m \cos \omega t) = \omega C V_m \cos \omega t$$

or $$i_C = I_m \sin(\omega t + 90°)$$

where $$I_m = \omega C V_m$$

Note that the peak value of i_C is directly related to $\omega\ (= 2\pi f)$ and C, as predicted in the discussion above.

A plot of v_C and i_C in Fig. 14.12 reveals that

for a capacitor, i_C leads v_C by 90°, or v_C lags i_C by 90°.*

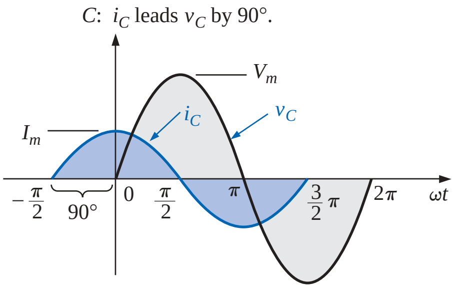

FIG. 14.12
The current of a purely capacitive element leads the voltage across the element by 90°.

If a phase angle is included in the sinusoidal expression for v_C, such as

$$v_C = V_m \sin(\omega t \pm \theta)$$

then $$i_C = \omega C V_m \sin(\omega t \pm \theta + 90°)$$

Applying

$$\text{Opposition} = \frac{\text{cause}}{\text{effect}}$$

and substituting values, we obtain

$$\text{Opposition} = \frac{V_m}{I_m} = \frac{V_m}{\omega C V_m} = \frac{1}{\omega C}$$

which agrees with the results obtained above.

The quantity $1/\omega C$, called the **reactance** of a capacitor, is symbolically represented by X_C and is measured in ohms; that is,

$$\boxed{X_C = \frac{1}{\omega C}} \qquad \text{(ohms, } \Omega) \qquad \textbf{(14.6)}$$

In an Ohm's law format, its magnitude can be determined from

$$\boxed{X_C = \frac{V_m}{I_m}} \qquad \text{(ohms, } \Omega) \qquad \textbf{(14.7)}$$

Capacitive reactance is the opposition to the flow of charge, which results in the continual interchange of energy between the source and the electric field of the capacitor. Like the inductor, the capacitor does *not* dissipate energy in any form (ignoring the effects of the leakage resistance).

In the circuits just considered, the current was given in the inductive circuit, and the voltage in the capacitive circuit. This was done to avoid the use of integration in finding the unknown quantities. In the inductive circuit,

$$v_L = L\frac{di_L}{dt}$$

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*A mnemonic phrase sometimes used to remember the phase relationship between the voltage and current of a coil and capacitor is "*ELI* the *ICE* man." Note that the *L* (inductor) has the *E* before the *I* (*e* leads *i* by 90°), and the *C* (capacitor) has the *I* before the *E* (*i* leads *e* by 90°).

but
$$\boxed{i_L = \frac{1}{L}\int v_L\,dt} \tag{14.8}$$

In the capacitive circuit,

$$i_C = C\,\frac{dv_C}{dt}$$

but
$$\boxed{v_C = \frac{1}{C}\int i_C\,dt} \tag{14.9}$$

Shortly, we shall consider a method of analyzing ac circuits that will permit us to solve for an unknown quantity with sinusoidal input without having to use direct integration or differentiation.

It is possible to determine whether a network with one or more elements is predominantly capacitive or inductive by noting the phase relationship between the input voltage and current.

If the source current leads the applied voltage, the network is predominantly capacitive, and if the applied voltage leads the source current, it is predominantly inductive.

Since we now have an equation for the reactance of an inductor or capacitor, we do not need to use derivatives or integration in the examples to be considered. Simply applying Ohm's law, $I_m = E_m/X_L$ (or X_C), and keeping in mind the phase relationship between the voltage and current for each element, will be sufficient to complete the examples.

EXAMPLE 14.1 The voltage across a resistor is indicated. Find the sinusoidal expression for the current if the resistor is 10 Ω. Sketch the curves for v and i.

a. $v = 100 \sin 377t$
b. $v = 25 \sin(377t + 60°)$

Solutions:

a. Eq. (14.2): $I_m = \dfrac{V_m}{R} = \dfrac{100\text{ V}}{10\ \Omega} = 10\text{ A}$

(v and i are in phase), resulting in

$$i = \mathbf{10 \sin 377}\boldsymbol{t}$$

The curves are sketched in Fig. 14.13.

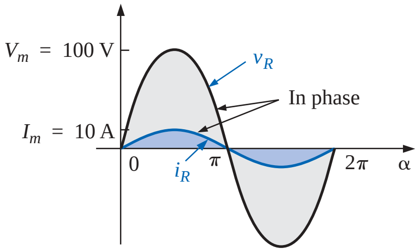

FIG. 14.13
Example 14.1(a).

b. Eq. (14.2): $I_m = \dfrac{V_m}{R} = \dfrac{25\text{ V}}{10\ \Omega} = 2.5\text{ A}$

(v and i are in phase), resulting in

$$i = \mathbf{2.5 \sin(377}\boldsymbol{t}\mathbf{ + 60°)}$$

The curves are sketched in Fig. 14.14.

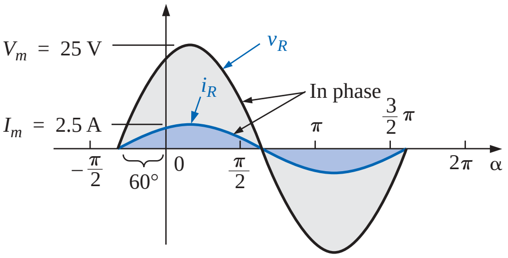

FIG. 14.14
Example 14.1(b).

EXAMPLE 14.2 The current through a 5-Ω resistor is given. Find the sinusoidal expression for the voltage across the resistor for $i = 40 \sin(377t + 30°)$.

Solution: Eq. (14.3): $V_m = I_mR = (40\text{ A})(5\ \Omega) = 200\text{ V}$

(v and i are in phase), resulting in

$$v = \mathbf{200 \sin(377}\boldsymbol{t}\mathbf{ + 30°)}$$

EXAMPLE 14.3 The current through a 0.1-H coil is provided. Find the sinusoidal expression for the voltage across the coil. Sketch the v and i curves.

a. $i = 10 \sin 377t$
b. $i = 7 \sin(377t - 70°)$

Solutions:

a. Eq. (14.4): $X_L = \omega L = (377\text{ rad/s})(0.1\text{ H}) = 37.7\ \Omega$
Eq. (14.5): $V_m = I_mX_L = (10\text{ A})(37.7\ \Omega) = 377\text{ V}$

and we know that for a coil v leads i by 90°. Therefore,

$$v = \mathbf{377 \sin(377}\boldsymbol{t}\mathbf{ + 90°)}$$

The curves are sketched in Fig. 14.15.

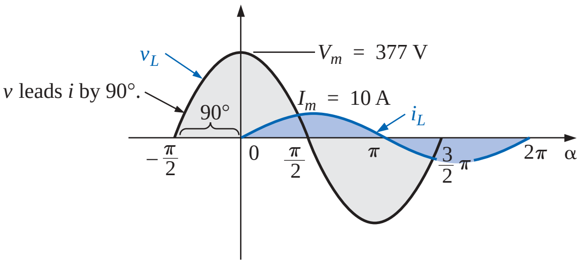

FIG. 14.15
Example 14.3(a).

b. X_L remains at 37.7 Ω.

$$V_m = I_m X_L = (7\text{ A})(37.7\ \Omega) = 263.9\text{ V}$$

and we know that for a coil v leads i by 90°. Therefore,

$$v = 263.9 \sin(377t - 70° + 90°)$$

and

$$v = \mathbf{263.9 \sin(377}\boldsymbol{t}\ \mathbf{+ 20°)}$$

The curves are sketched in Fig. 14.16.

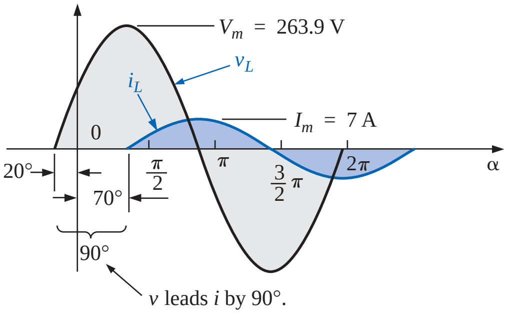

FIG. 14.16

Example 14.3(b).

EXAMPLE 14.4 The voltage across a 0.5-H coil is provided below. What is the sinusoidal expression for the current?

$$v = 100 \sin 20t$$

Solution:

$$X_L = \omega L = (20\text{ rad/s})(0.5\text{ H}) = 10\ \Omega$$

$$I_m = \frac{V_m}{X_L} = \frac{100\text{ V}}{10\ \Omega} = 10\text{ A}$$

and we know that i lags v by 90°. Therefore,

$$i = \mathbf{10 \sin(20}\boldsymbol{t}\ \mathbf{- 90°)}$$

EXAMPLE 14.5 The voltage across a 1-μF capacitor is provided below. What is the sinusoidal expression for the current? Sketch the v and i curves.

$$v = 30 \sin 400t$$

Solution:

Eq. (14.6): $$X_C = \frac{1}{\omega C} = \frac{1}{(400\text{ rad/s})(1 \times 10^{-6}\text{ F})} = \frac{10^6\ \Omega}{400} = 2500\ \Omega$$

Eq. (14.7): $$I_m = \frac{V_m}{X_C} = \frac{30\text{ V}}{2500\ \Omega} = 0.0120\text{ A} = 12\text{ mA}$$

and we know that for a capacitor i leads v by 90°. Therefore,

$$i = \mathbf{12 \times 10^{-3} \sin(400}\boldsymbol{t}\ \mathbf{+ 90°)}$$

The curves are sketched in Fig. 14.17.

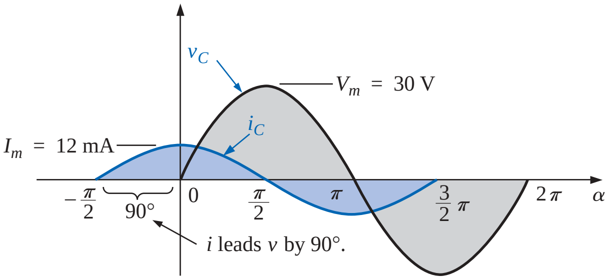

FIG. 14.17
Example 14.5.

EXAMPLE 14.6 The current through a 100-μF capacitor is given. Find the sinusoidal expression for the voltage across the capacitor.

$$i = 40 \sin(500t + 60°)$$

Solution:

$$X_C = \frac{1}{\omega C} = \frac{1}{(500 \text{ rad/s})(100 \times 10^{-6} \text{ F})} = \frac{10^6 \ \Omega}{5 \times 10^4} = \frac{10^2 \ \Omega}{5} = 20 \ \Omega$$

$$V_m = I_m X_C = (40 \text{ A})(20 \ \Omega) = 800 \text{ V}$$

and we know that for a capacitor, *v* lags *i* by 90°. Therefore,

$$v = 800 \sin(500t + 60° - 90°)$$

and $$v = \mathbf{800 \sin(500\boldsymbol{t} - 30°)}$$

EXAMPLE 14.7 For the following pairs of voltages and currents, determine whether the element involved is a capacitor, an inductor, or a resistor, and determine the value of *C*, *L*, or *R* if sufficient data are provided (Fig. 14.18):

a. $v = 100 \sin(\omega t + 40°)$
   $i = 20 \sin(\omega t + 40°)$
b. $v = 1000 \sin(377t + 10°)$
   $i = 5 \sin(377t - 80°)$
c. $v = 500 \sin(157t + 30°)$
   $i = 1 \sin(157t + 120°)$
d. $v = 50 \cos(\omega t + 20°)$
   $i = 5 \sin(\omega t + 110°)$

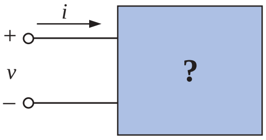

FIG. 14.18
Example 14.7.

Solutions:

a. Since *v* and *i* are *in phase*, the element is a *resistor*, and

$$R = \frac{V_m}{I_m} = \frac{100 \text{ V}}{20 \text{ A}} = \mathbf{5 \ \Omega}$$

b. Since *v leads i* by 90°, the element is an *inductor*, and

$$X_L = \frac{V_m}{I_m} = \frac{1000 \text{ V}}{5 \text{ A}} = 200 \ \Omega$$

so that $X_L = \omega L = 200 \ \Omega$ or

$$L = \frac{200\ \Omega}{\omega} = \frac{200\ \Omega}{377\ \text{rad/s}} = \mathbf{0.531\ H}$$

c. Since i *leads* v by 90°, the element is a *capacitor,* and

$$X_C = \frac{V_m}{I_m} = \frac{500\ \text{V}}{1\ \text{A}} = 500\ \Omega$$

so that $X_C = \dfrac{1}{\omega C} = 500\ \Omega$ or

$$C = \frac{1}{\omega 500\ \Omega} = \frac{1}{(157\ \text{rad/s})(500\ \Omega)} = \mathbf{12.74\ \mu F}$$

d. $v = 50\cos(\omega t + 20°) = 50\sin(\omega t + 20° + 90°)$
$\quad = 50\sin(\omega t + 110°)$

Since v and i are *in phase,* the element is a *resistor,* and

$$R = \frac{V_m}{I_m} = \frac{50\ \text{V}}{5\ \text{A}} = \mathbf{10\ \Omega}$$

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dc, High-, and Low-Frequency Effects on *L* and *C*

For dc circuits, the frequency is zero, and the reactance of a coil is

$$X_L = 2\pi f L = 2\pi(0)L = 0\ \Omega$$

The use of the short-circuit equivalence for the inductor in dc circuits (Chapter 12) is now validated. At very high frequencies, $X_L\!\uparrow\ = 2\pi f\!\uparrow\! L$ is very large, and for some practical applications the inductor can be replaced by an open circuit. In equation form,

$$\boxed{X_L = 0\ \Omega} \quad \text{dc, } f = 0\ \text{Hz} \qquad \textbf{(14.10)}$$

and

$$\boxed{X_L \Rightarrow \infty\ \Omega \quad \text{as } f \Rightarrow \infty\ \text{Hz}} \qquad \textbf{(14.11)}$$

The capacitor can be replaced by an open-circuit equivalence in dc circuits since $f = 0$, and

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi(0)C} \Rightarrow \infty\ \Omega$$

once again substantiating our previous action (Chapter 10). At very high frequencies, for finite capacitances,

$$X_C\!\downarrow\ = \frac{1}{2\pi f\!\uparrow\! C}$$

is very small, and for some practical applications the capacitor can be replaced by a short circuit. In equation form

$$\boxed{X_C \Rightarrow \infty\ \Omega \quad \text{as } f \Rightarrow 0\ \text{Hz}} \qquad \textbf{(14.12)}$$

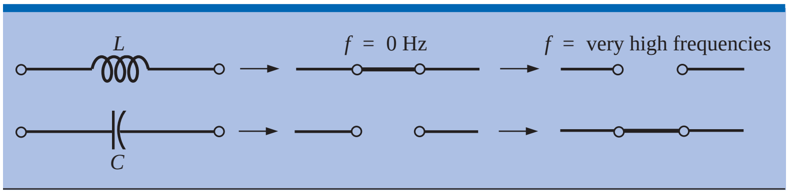
and

$$\boxed{X_C \cong 0\ \Omega} \quad f = \text{very high frequencies} \qquad \textbf{(14.13)}$$

Table 14.1 reviews the preceding conclusions.

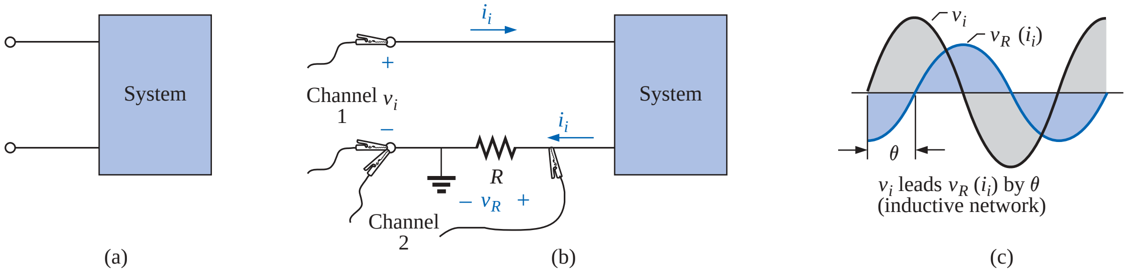
TABLE 14.1

Effect of high and low frequencies on the circuit model of an inductor and a capacitor.



Phase Angle Measurements between the Applied Voltage and Source Current

Now that we are familiar with phase relationships and understand how the elements affect the phase relationship between the applied voltage and resulting current, the use of the oscilloscope to measure the phase angle can be introduced. Recall from past discussions that the oscilloscope can be used only to display voltage levels versus time. However, now that we realize that the voltage across a resistor is in phase with the current through a resistor, we can consider the phase angle associated with the voltage across any resistor actually to be the phase angle of the current. For example, suppose that we want to find the phase angle introduced by the unknown system of Fig. 14.19(a). In Fig. 14.19(b), a resistor was added to the input leads, and the two channels of a dual trace (most modern-day oscilloscopes can display two signals at the same time) were connected as shown. One channel will display the input voltage v_i, whereas the other will display v_R, as shown in Fig. 14.19(c). However, as noted before, since v_R and i_R are in phase, the phase angle appearing in Fig. 14.19(c) is also the phase angle between v_i and i_i. The addition of a "sensing" resistor (a resistor of a magnitude that will not adversely affect the input characteristics of the system), therefore, can be used to determine the phase angle introduced by the system and can be used to determine the magnitude of the resulting current. The details of the connections that must be made and how the actual phase angle is determined will be left for the laboratory experience.



(a) (b) (c)

FIG. 14.19

Using an oscilloscope to determine the phase angle between the applied voltage and the source current.

14.4 FREQUENCY RESPONSE OF THE BASIC ELEMENTS

The analysis of Section 14.3 was limited to a particular applied frequency. What is the effect of varying the frequency on the level of opposition offered by a resistive, inductive, or capacitive element? We are aware from the last section that the inductive reactance increases with frequency while the capacitive reactance decreases. However, what is the pattern to this increase or decrease in opposition? Does it continue indefinitely on the same path? Since applied signals may have frequencies extending from a few hertz to megahertz, it is important to be aware of the effect of frequency on the opposition level.

R

Thus far we have assumed that the resistance of a resistor is independent of the applied frequency. However, in the real world each resistive element has stray capacitance levels and lead inductance that are sensitive to the applied frequency. However, the capacitive and inductive levels involved are usually so small that their real effect is not noticed until the megahertz range. The resistance-versus-frequency curves for a number of carbon composition resistors are provided in Fig. 14.20. Note that the lower resistance levels seem to be less affected by the frequency level. The 100-Ω resistor is essentially stable up to about 300 MHz, whereas the 100-kΩ resistor starts its radical decline at about 15 MHz.

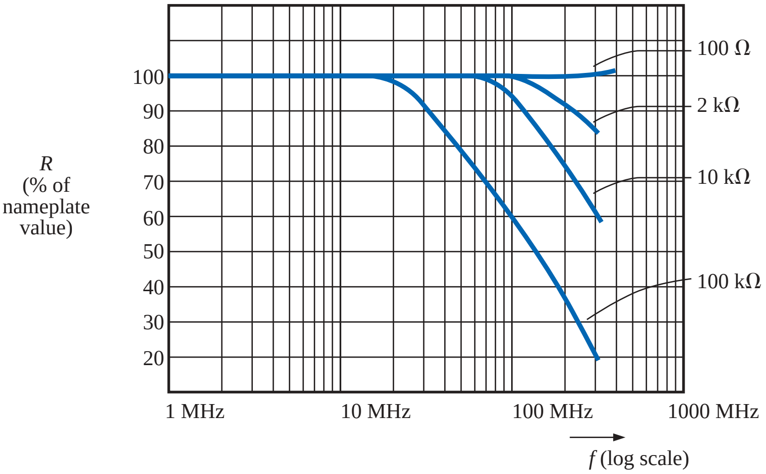


FIG. 14.20

Typical resistance-versus-frequency curves for carbon compound resistors.

Frequency, therefore, does have impact on the resistance of an element, but for our current frequency range of interest, we will assume the resistance-versus-frequency plot of Fig. 14.21 (like Fig. 14.20 up to 15 MHz), which essentially specifies that the resistance level of a resistor is independent of frequency.

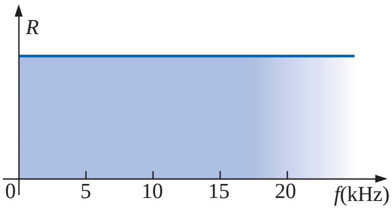


FIG. 14.21

R versus f for the range of interest.

L

For inductors, the equation

$$X_L = \omega L = 2\pi f L = 2\pi L f$$

is directly related to the straight-line equation

$$y = mx + b = (2\pi L)f + 0$$

with a slope (m) of $2\pi L$ and a y-intercept (b) of zero. X_L is the y variable and f is the x variable, as shown in Fig. 14.22.

The larger the inductance, the greater the slope ($m = 2\pi L$) for the same frequency range, as shown in Fig. 14.22. Keep in mind, as reemphasized by Fig. 14.22, that the opposition of an inductor at very low frequencies approaches that of a short circuit, while at high frequencies the reactance approaches that of an open circuit.

For the capacitor, the reactance equation

$$X_C = \frac{1}{2\pi fC}$$

can be written

$$X_C f = \frac{1}{2\pi C}$$

which matches the basic format of a hyperbola,

$$yx = k$$

with $y = X_C$, $x = f$, and the constant $k = 1/(2\pi C)$.

At $f = 0$ Hz, the reactance of the capacitor is so large, as shown in Fig. 14.23, that it can be replaced by an open-circuit equivalent. As the frequency increases, the reactance decreases, until eventually a short-circuit equivalent would be appropriate. Note that an increase in capacitance causes the reactance to drop off more rapidly with frequency.

In summary, therefore, as the applied frequency increases, the resistance of a resistor remains constant, the reactance of an inductor increases linearly, and the reactance of a capacitor decreases nonlinearly.

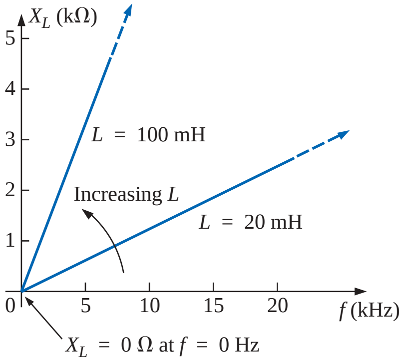


FIG. 14.22

X_L versus frequency.

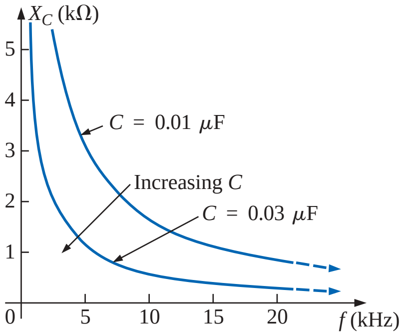


FIG. 14.23

X_C versus frequency.

EXAMPLE 14.8 At what frequency will the reactance of a 200-mH inductor match the resistance level of a 5-kΩ resistor?

Solution: The resistance remains constant at 5 kΩ for the frequency range of the inductor. Therefore,

$$\begin{aligned} R = 5000\ \Omega = X_L &= 2\pi fL = 2\pi Lf \\ &= 2\pi(200 \times 10^{-3}\ \text{H})f = 1.257f \end{aligned}$$

and

$$f = \frac{5000\ \text{Hz}}{1.257} \cong \mathbf{3.98\ kHz}$$

EXAMPLE 14.9 At what frequency will an inductor of 5 mH have the same reactance as a capacitor of 0.1 μF?

Solution:

$$X_L = X_C$$

$$2\pi fL = \frac{1}{2\pi fC}$$

$$f^2 = \frac{1}{4\pi^2 LC}$$

and

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(5 \times 10^{-3}\ \text{H})(0.1 \times 10^{-6}\ \text{F})}}$$

$$= \frac{1}{2\pi\sqrt{5 \times 10^{-10}}} = \frac{1}{(2\pi)(2.236 \times 10^{-5})}$$

$$f = \frac{10^5\ \text{Hz}}{14.05} \cong \mathbf{7.12\ kHz}$$

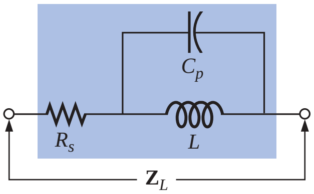


FIG. 14.24
Practical equivalent for an inductor.

One must also be aware that commercial inductors are not ideal elements. In other words, the terminal characteristics of an inductance will vary with several factors, such as frequency, temperature, and current. A true equivalent for an inductor appears in Fig. 14.24. The series resistance R_s represents the copper losses (resistance of the many turns of thin copper wire); the eddy current losses (which will be described in Chapter 19 and which are losses due to small circular currents in the core when an ac voltage is applied); and the hysteresis losses (which will also be described in Chapter 19 and which are losses due to core losses created by the rapidly reversing field in the core). The capacitance C_p is the stray capacitance that exists between the windings of the inductor. For most inductors, the construction is usually such that the larger the inductance, the lower the frequency at which the parasitic elements become important. That is, for inductors in the millihenry range (which is very typical), frequencies approaching 100 kHz can have an effect on the ideal characteristics of the element. For inductors in the microhenry range, a frequency of 1 MHz may introduce negative effects. This is not to suggest that the inductors lose their effect at these frequencies but more that they can no longer be considered ideal (purely inductive elements).

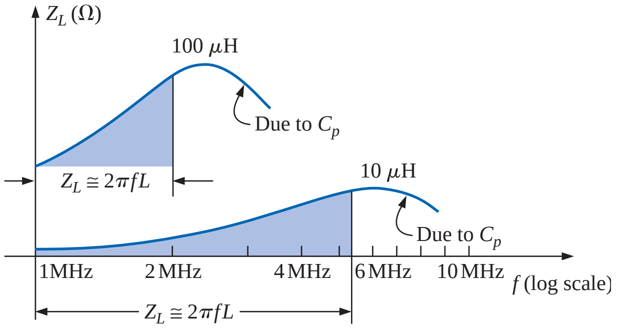


FIG. 14.25
Z_L versus frequency for the practical inductor equivalent of Fig. 14.24.

Figure 14.25 is a plot of the magnitude of the impedance Z_L of Fig. 14.24 versus frequency. Note that up to about 2 MHz the impedance increases almost linearly with frequency, clearly suggesting that the 100-μH inductor is essentially ideal. However, above 2 MHz all the factors contributing to R_s will start to increase, while the reactance due to the capacitive element C_p will be more pronounced. The dropping level of capacitive reactance will begin to have a shorting effect across the windings of the inductor and will reduce the overall inductive effect. Eventually, if the frequency continues to increase, the capacitive effects will overcome the inductive effects, and the element will actually begin to behave in a capacitive fashion. Note the similarities of this region with the curves of Fig. 14.23. Also note that decreasing levels of inductance (available with fewer turns and therefore lower levels of C_p) will not demonstrate the degrading effect until higher frequencies are applied. In general, therefore, the frequency of application for a coil becomes important at increasing frequencies. Inductors lose their ideal characteristics and in fact begin to act as capacitive elements with increasing losses at very high frequencies.

The capacitor, like the inductor, is not ideal at higher frequencies. In fact, a transition point can be defined where the characteristics of the capacitor will actually be inductive. The complete equivalent model for a capacitor is provided in Fig. 14.26. The resistance R_s, defined by the resistivity of the dielectric (typically 10^{12} $\Omega \cdot$m or better) and the case resistance, will determine the level of leakage current to expect during the discharge cycle. In other words, a charged capacitor can discharge both through the case and through the dielectric at a rate determined by the resistance of each path. Depending on the capacitor, the discharge time can extend from a few seconds for some electrolytic capacitors to hours (paper) or perhaps days (polystyrene). Inversely, therefore, electrolytics obviously have much lower levels of R_s than paper or polystyrene. The resistance R_p reflects the energy lost as the atoms continually realign themselves in the dielectric due to the applied alternating ac voltage. Molecular friction is present due to the motion of the atoms as they respond to the alternating applied electric field. Interestingly enough, however, the relative permittivity will decrease with increasing frequencies but will eventually take a complete turnaround and begin to increase at very high frequencies. The inductance L_s includes the inductance of the capacitor leads and any inductive effects introduced by the design of the capacitor. Be aware that the inductance of the leads is about 0.05 μH per centimeter or 0.2 μH for a capacitor with two 2-cm leads—a level that can be important at high frequencies. As for the inductor, the capacitor will behave quite ideally for the low- and mid-frequency range, as shown by the plot of Fig. 14.27 for a 0.01-μF metalized film capacitor with 2-cm leads. As the frequency increases, however, and the reactance X_s becomes larger, a frequency will eventually be reached where the reactance of the coil equals that of the capacitor (a resonant condition to be described in Chapter 20). Any additional increase in frequency will simply result in X_s being greater than X_C, and the element will behave like an inductor. In general, therefore, the frequency of application is important for capacitive elements because

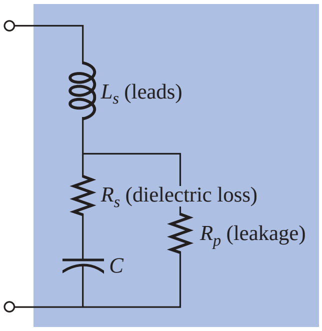


FIG. 14.26
Practical equivalent for a capacitor.

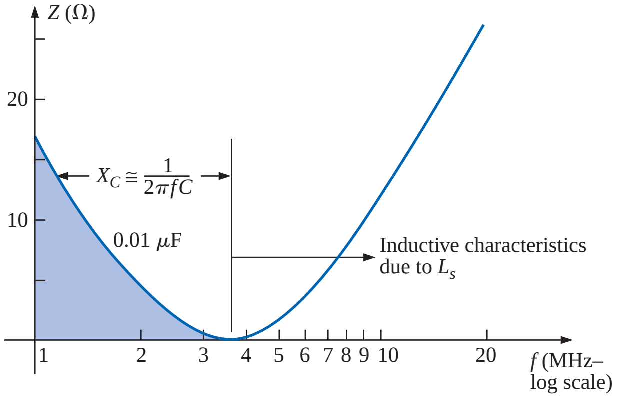


FIG. 14.27
Impedance characteristics of a 0.01-μF metalized film capacitor versus frequency.

there comes a point with increasing frequency when the element will take on inductive characteristics. It also points out that the frequency of application defines the type of capacitor (or inductor) that would be applied: Electrolytics are limited to frequencies up to perhaps 10 kHz, while ceramic or mica can handle frequencies beyond 10 MHz.

The expected temperature range of operation can have an important impact on the type of capacitor chosen for a particular application. Electrolytics, tantalum, and some high-*k* ceramic capacitors are very sensitive to colder temperatures. In fact, most electrolytics lose 20% of their room-temperature capacitance at 0°C (freezing). Higher temperatures (up to 100°C or 212°F) seem to have less of an impact in general than colder temperatures, but high-*k* ceramics can lose up to 30% of their capacitance level at 100°C compared to room temperature. With exposure and experience, you will learn the type of capacitor employed for each application, and concern will arise only when very high frequencies, extreme temperatures, or very high currents or voltages are encountered.

14.5 AVERAGE POWER AND POWER FACTOR

For any load in a sinusoidal ac network, the voltage across the load and the current through the load will vary in a sinusoidal nature. The questions then arise, How does the power to the load determined by the product $v \cdot i$ vary, and what fixed value can be assigned to the power since it will vary with time?

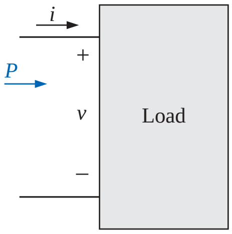


FIG. 14.28
Determining the power delivered in a sinusoidal ac network.

If we take the general case depicted in Fig. 14.28 and use the following for v and i:

$$v = V_m \sin(\omega t + \theta_v)$$
$$i = I_m \sin(\omega t + \theta_i)$$

then the power is defined by

$$\begin{aligned} p = vi &= V_m \sin(\omega t + \theta_v) I_m \sin(\omega t + \theta_i) \\ &= V_m I_m \sin(\omega t + \theta_v)\sin(\omega t + \theta_i) \end{aligned}$$

Using the trigonometric identity

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$

the function $\sin(\omega t + \theta_v)\sin(\omega t + \theta_i)$ becomes

$$\begin{aligned} &\sin(\omega t + \theta_v)\sin(\omega t + \theta_i) \\ &= \frac{\cos[(\omega t + \theta_v) - (\omega t + \theta_i)] - \cos[(\omega t + \theta_v) + (\omega t + \theta_i)]}{2} \\ &= \frac{\cos(\theta_v - \theta_i) - \cos(2\omega t + \theta_v + \theta_i)}{2} \end{aligned}$$

so that

$$p = \overbrace{\left[\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\right]}^{\text{Fixed value}} - \overbrace{\left[\frac{V_m I_m}{2}\cos(2\omega t + \theta_v + \theta_i)\right]}^{\text{Time-varying (function of } t)}$$

A plot of v, i, and p on the same set of axes is shown in Fig. 14.29.

Note that the second factor in the preceding equation is a cosine wave with an amplitude of $V_m I_m/2$ and with a frequency twice that of

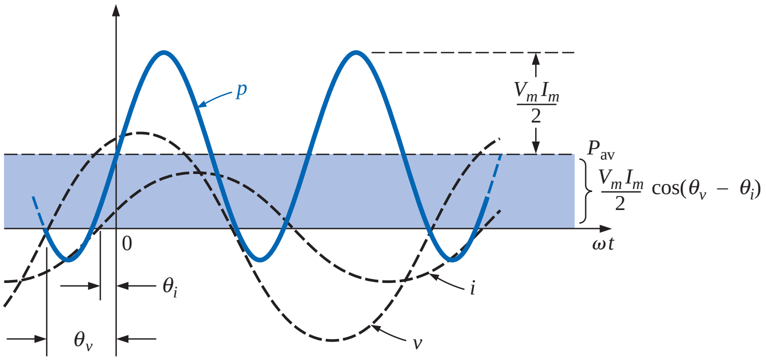


FIG. 14.29

Defining the average power for a sinusoidal ac network.

the voltage or current. The average value of this term is zero over one cycle, producing no net transfer of energy in any one direction.

The first term in the preceding equation, however, has a constant magnitude (no time dependence) and therefore provides some net transfer of energy. This term is referred to as the **average power,** the reason for which is obvious from Fig. 14.29. The average power, or **real power** as it is sometimes called, is the power delivered to and dissipated by the load. It corresponds to the power calculations performed for dc networks. The angle $(\theta_v - \theta_i)$ is the phase angle between v and i. Since $\cos(-\alpha) = \cos\alpha$,

the magnitude of average power delivered is independent of whether v leads i or i leads v.

Defining θ as equal to $|\theta_v - \theta_i|$, where $|\quad|$ indicates that only the magnitude is important and the sign is immaterial, we have

$$\boxed{P = \frac{V_m I_m}{2}\cos\theta} \qquad \text{(watts, W)} \qquad \textbf{(14.14)}$$

where P is the average power in watts. This equation can also be written

$$P = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta$$

or, since $$V_{\text{eff}} = \frac{V_m}{\sqrt{2}} \quad \text{and} \quad I_{\text{eff}} = \frac{I_m}{\sqrt{2}}$$

Equation (14.14) becomes

$$\boxed{P = V_{\text{eff}} I_{\text{eff}} \cos\theta} \qquad \textbf{(14.15)}$$

Let us now apply Eqs. (14.14) and (14.15) to the basic R, L, and C elements.

Resistor

In a purely resistive circuit, since v and i are in phase, $|\theta_v - \theta_i| = \theta = 0°$, and $\cos\theta = \cos 0° = 1$, so that

$$\boxed{P = \frac{V_m I_m}{2} = V_{\text{eff}} I_{\text{eff}}} \qquad \text{(W)} \qquad \textbf{(14.16)}$$

Or, since

$$I_{\text{eff}} = \frac{V_{\text{eff}}}{R}$$

then

$$\boxed{P = \frac{V_{\text{eff}}^2}{R} = I_{\text{eff}}^2 R} \qquad \text{(W)} \qquad \textbf{(14.17)}$$

Inductor

In a purely inductive circuit, since v leads i by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2} \cos 90° = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal inductor (no associated resistance) is zero watts.

Capacitor

In a purely capacitive circuit, since i leads v by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2} \cos(90°) = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal capacitor (no associated resistance) is zero watts.

EXAMPLE 14.10 Find the average power dissipated in a network whose input current and voltage are the following:

$$i = 5 \sin(\omega t + 40°)$$
$$v = 10 \sin(\omega t + 40°)$$

Solution: Since v and i are in phase, the circuit appears to be purely resistive at the input terminals. Therefore,

$$P = \frac{V_m I_m}{2} = \frac{(10\text{ V})(5\text{ A})}{2} = \mathbf{25\ W}$$

or

$$R = \frac{V_m}{I_m} = \frac{10\text{ V}}{5\text{ A}} = 2\ \Omega$$

and

$$P = \frac{V_{\text{eff}}^2}{R} = \frac{[(0.707)(10\text{ V})]^2}{2} = \mathbf{25\ W}$$

or

$$P = I_{\text{eff}}^2 R = [(0.707)(5\text{ A})]^2(2) = \mathbf{25\ W}$$

For the following example, the circuit consists of a combination of resistances and reactances producing phase angles between the input current and voltage different from 0° or 90°.

EXAMPLE 14.11 Determine the average power delivered to networks having the following input voltage and current:

a. $v = 100 \sin(\omega t + 40°)$
$i = 20 \sin(\omega t + 70°)$

b. $v = 150 \sin(\omega t - 70°)$
$i = 3 \sin(\omega t - 50°)$

Solutions:

a. $V_m = 100, \quad \theta_v = 40°$
$I_m = 20, \quad \theta_i = 70°$
$\theta = |\theta_v - \theta_i| = |40° - 70°| = |-30°| = 30°$
and

$$P = \frac{V_m I_m}{2} \cos\theta = \frac{(100\text{ V})(20\text{ A})}{2} \cos(30°) = (1000\text{ W})(0.866)$$
$$= \mathbf{866\ W}$$

b. $V_m = 150\text{ V}, \quad \theta_v = -70°$
$I_m = 3\text{ A}, \quad \theta_i = -50°$
$\theta = |\theta_v - \theta_i| = |-70° - (-50°)|$
$= |-70° + 50°| = |-20°| = 20°$
and

$$P = \frac{V_m I_m}{2} \cos\theta = \frac{(150\text{ V})(3\text{ A})}{2} \cos(20°) = (225\text{ W})(0.9397)$$
$$= \mathbf{211.43\ W}$$

Power Factor

In the equation $P = (V_m I_m/2)\cos\theta$, the factor that has significant control over the delivered power level is the $\cos\theta$. No matter how large the voltage or current, if $\cos\theta = 0$, the power is zero; if $\cos\theta = 1$, the power delivered is a maximum. Since it has such control, the expression was given the name **power factor** and is defined by

$$\boxed{\text{Power factor} = F_p = \cos\theta} \qquad \textbf{(14.18)}$$

For a purely resistive load such as the one shown in Fig. 14.30, the phase angle between v and i is 0° and $F_p = \cos\theta = \cos 0° = 1$. The power delivered is a maximum of $(V_m I_m/2)\cos\theta = ((100\text{ V})(5\text{ A})/2) \cdot (1) = 250\text{ W}$.

For a purely reactive load (inductive or capacitive) such as the one shown in Fig. 14.31, the phase angle between v and i is 90° and $F_p = \cos\theta = \cos 90° = 0$. The power delivered is then the minimum value of zero watts, **even though the current has the same peak value** as that encountered in Fig. 14.30.

For situations where the load is a combination of resistive and reactive elements, the power factor will vary between 0 and 1. The more resistive the total impedance, the closer the power factor is to 1; the more reactive the total impedance, the closer the power factor is to 0.

In terms of the average power and the terminal voltage and current,

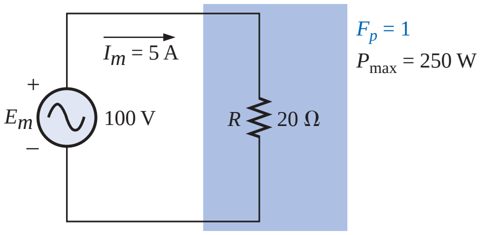


FIG. 14.30
Purely resistive load with $F_p = 1$.

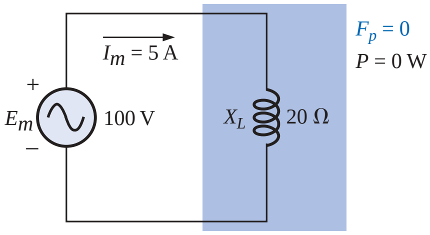


FIG. 14.31
Purely inductive load with $F_p = 0$.

$$\boxed{F_p = \cos\theta = \frac{P}{V_{\text{eff}}I_{\text{eff}}}} \qquad \textbf{(14.19)}$$

The terms *leading* and *lagging* are often written in conjunction with the power factor. *They are defined by the current through the load.* If the current leads the voltage across a load, the load has a **leading power factor.** If the current lags the voltage across the load, the load has a **lagging power factor.** In other words,

capacitive networks have leading power factors, and inductive networks have lagging power factors.

The importance of the power factor to power distribution systems is examined in Chapter 19. In fact, one section is devoted to power-factor correction.

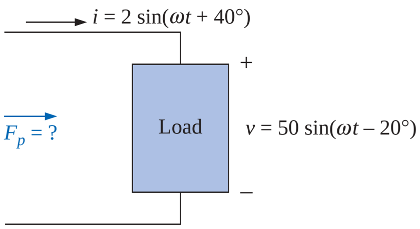


FIG. 14.32
Example 14.12(a).

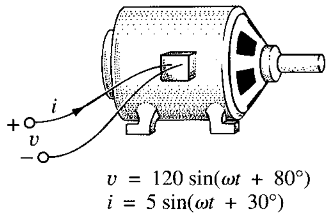


FIG. 14.33
Example 14.12(b).

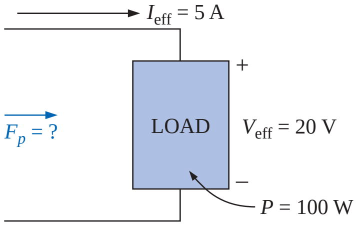


FIG. 14.34
Example 14.12(c).

EXAMPLE 14.12 Determine the power factors of the following loads, and indicate whether they are leading or lagging:

a. Fig. 14.32
b. Fig. 14.33
c. Fig. 14.34

Solutions:

a. $F_p = \cos\theta = \cos|40° - (-20°)| = \cos 60° = \mathbf{0.5\ leading}$

b. $F_p = \cos\theta\, |80° - 30°| = \cos 50° = \mathbf{0.6428\ lagging}$

c. $F_p = \cos\theta = \dfrac{P}{V_{\text{eff}}I_{\text{eff}}} = \dfrac{100\text{ W}}{(20\text{ V})(5\text{ A})} = \dfrac{100\text{ W}}{100\text{ W}} = \mathbf{1}$

The load is resistive, and F_p is neither leading nor lagging.

14.6 COMPLEX NUMBERS

In our analysis of dc networks, we found it necessary to determine the algebraic sum of voltages and currents. Since the same will also be true for ac networks, the question arises, How do we determine the algebraic sum of two or more voltages (or currents) that are varying sinusoidally? Although one solution would be to find the algebraic sum on a point-to-point basis (as shown in Section 14.12), this would be a long and tedious process in which accuracy would be directly related to the scale employed.

It is the purpose of this chapter to introduce a system of **complex numbers** that, when related to the sinusoidal ac waveform, will result in a technique for finding the algebraic sum of sinusoidal waveforms that is quick, direct, and accurate. In the following chapters, the technique will be extended to permit the analysis of sinusoidal ac networks in a manner very similar to that applied to dc networks. The methods and theorems as described for dc networks can then be applied to sinusoidal ac networks with little difficulty.

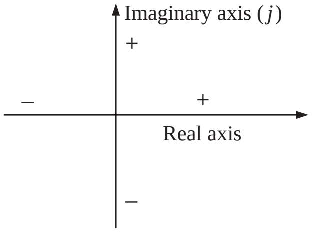


FIG. 14.35
Defining the real and imaginary axes of a complex plane.

A **complex number** represents a point in a two-dimensional plane located with reference to two distinct axes. This point can also determine a radius vector drawn from the origin to the point. The horizontal axis is called the *real* axis, while the vertical axis is called the *imaginary* axis. Both are labeled in Fig. 14.35. Every number from zero to $\pm\infty$ can be represented by some point along the real axis. Prior to the development of this system of complex numbers, it was believed that any number not on the real axis would not exist—hence the term *imaginary* for the vertical axis.

In the complex plane, the horizontal or real axis represents all positive numbers to the right of the imaginary axis and all negative numbers to the left of the imaginary axis. All positive imaginary numbers are represented above the real axis, and all negative imaginary numbers, below the real axis. The symbol *j* (or sometimes *i*) is used to denote the imaginary component.

Two forms are used to represent a complex number: **rectangular** and **polar.** Each can represent a point in the plane or a radius vector drawn from the origin to that point.

14.7 RECTANGULAR FORM

The format for the **rectangular form** is

$$\boxed{\mathbf{C} = X + jY} \qquad \textbf{(14.20)}$$

as shown in Fig. 14.36. The letter **C** was chosen from the word "complex." The **boldface** notation is for any number with magnitude and direction. The *italic* is for magnitude only.

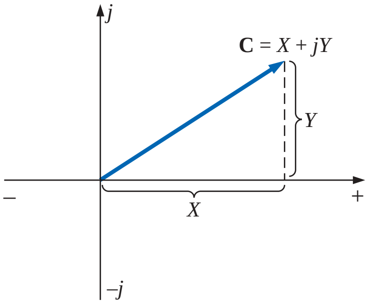


FIG. 14.36
Defining the rectangular form.

EXAMPLE 14.13 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 3 + j4$
b. $\mathbf{C} = 0 - j6$
c. $\mathbf{C} = -10 - j20$

Solutions:

a. See Fig. 14.37.
b. See Fig. 14.38.
c. See Fig. 14.39.

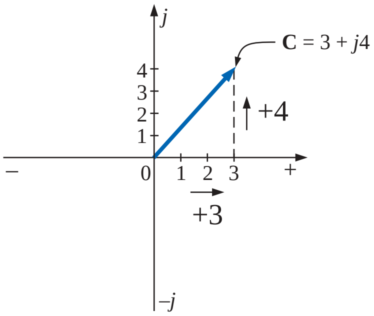


FIG. 14.37
Example 14.13(a).

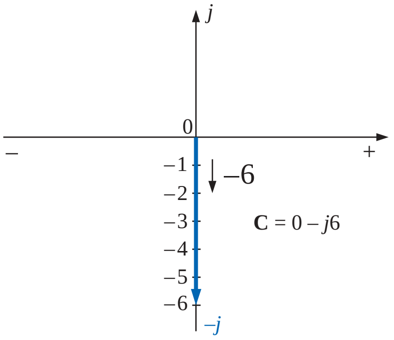


FIG. 14.38
Example 14.13(b).

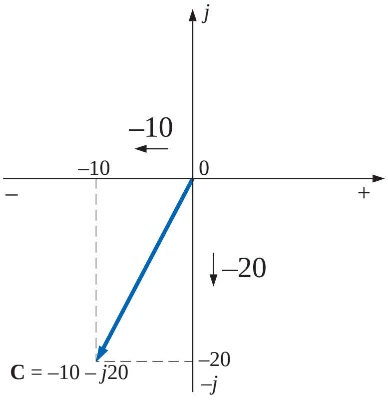


FIG. 14.39
Example 14.13(c).

14.8 POLAR FORM

The format for the **polar form** is

$$\boxed{\mathbf{C} = Z \angle \theta} \qquad \textbf{(14.21)}$$

with the letter *Z* chosen from the sequence *X*, *Y*, *Z*.

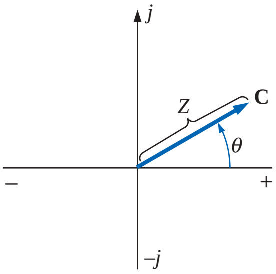


FIG. 14.40
Defining the polar form.

where Z indicates magnitude only and θ is **always measured counterclockwise (CCW) from the *positive real axis*,** as shown in Fig. 14.40. Angles measured in the clockwise direction from the positive real axis must have a negative sign associated with them.

A negative sign in front of the polar form has the effect shown in Fig. 14.41. Note that it results in a complex number directly opposite the complex number with a positive sign.

$$\boxed{-\mathbf{C} = -Z\angle\theta = Z\angle\theta \pm 180^\circ} \qquad \textbf{(14.22)}$$

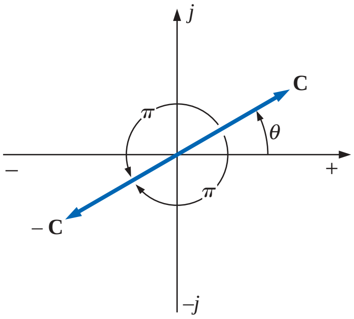


FIG. 14.41
Demonstrating the effect of a negative sign on the polar form.

EXAMPLE 14.14 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 5\angle 30^\circ$
b. $\mathbf{C} = 7\angle -120^\circ$
c. $\mathbf{C} = -4.2\angle 60^\circ$

Solutions:

a. See Fig. 14.42.
b. See Fig. 14.43.
c. See Fig. 14.44.

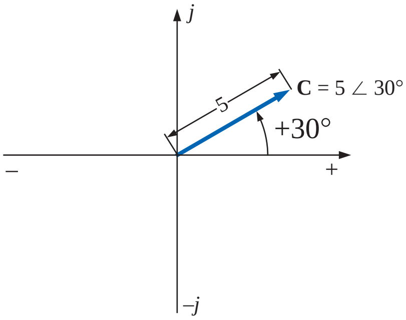


FIG. 14.42
Example 14.14(a).

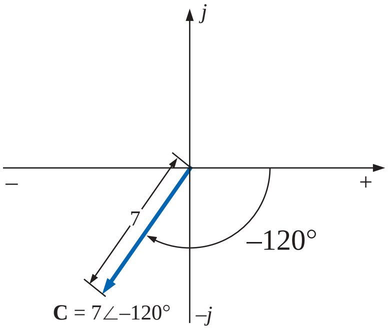


FIG. 14.43
Example 14.14(b).

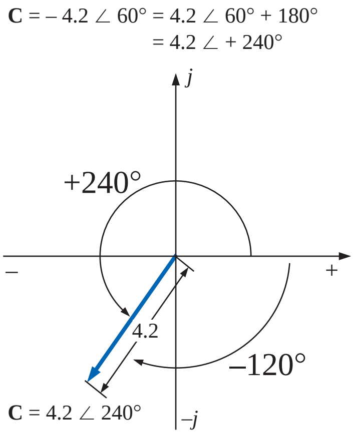


FIG. 14.44
Example 14.14(c).

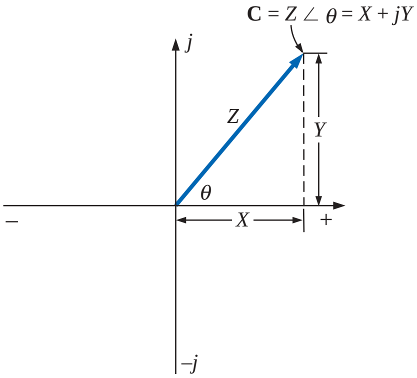


FIG. 14.45
Conversion between forms.

14.9 CONVERSION BETWEEN FORMS

The two forms are related by the following equations, as illustrated in Fig. 14.45.

Rectangular to Polar

$$\boxed{Z = \sqrt{X^2 + Y^2}} \qquad \textbf{(14.23)}$$

$$\boxed{\theta = \tan^{-1}\frac{Y}{X}} \qquad \textbf{(14.24)}$$

Polar to Rectangular

$$\boxed{X = Z\cos\theta} \qquad (14.25)$$

$$\boxed{Y = Z\sin\theta} \qquad (14.26)$$

EXAMPLE 14.15 Convert the following from rectangular to polar form:

$$\mathbf{C} = 3 + j\,4 \qquad \text{(Fig. 14.46)}$$

Solution:

$$Z = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13°$$

and $$\mathbf{C = 5\,\angle 53.13°}$$

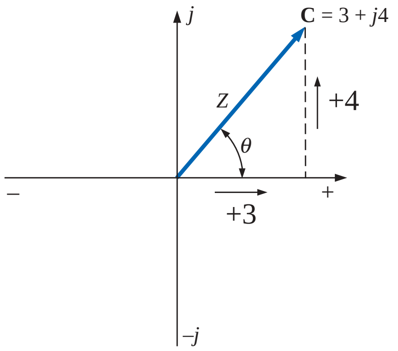


FIG. 14.46
Example 14.15.

EXAMPLE 14.16 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10\,\angle 45° \qquad \text{(Fig. 14.47)}$$

Solution:

$$X = 10\cos 45° = (10)(0.707) = 7.07$$
$$Y = 10\sin 45° = (10)(0.707) = 7.07$$

and $$\mathbf{C = 7.07 + \mathit{j}7.07}$$

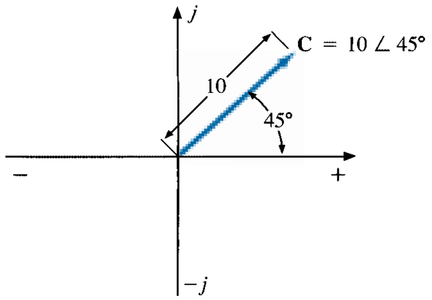


FIG. 14.47
Example 14.16.

If the complex number should appear in the second, third, or fourth quadrant, simply convert it in that quadrant, and carefully determine the proper angle to be associated with the magnitude of the vector.

EXAMPLE 14.17 Convert the following from rectangular to polar form:

$$\mathbf{C} = -6 + j\,3 \qquad \text{(Fig. 14.48)}$$

Solution:

$$Z = \sqrt{(6)^2 + (3)^2} = \sqrt{45} = 6.71$$

$$\beta = \tan^{-1}\left(\frac{3}{6}\right) = 26.57°$$

$$\theta = 180° - 26.57° = 153.43°$$

and $$\mathbf{C = 6.71\,\angle 153.43°}$$

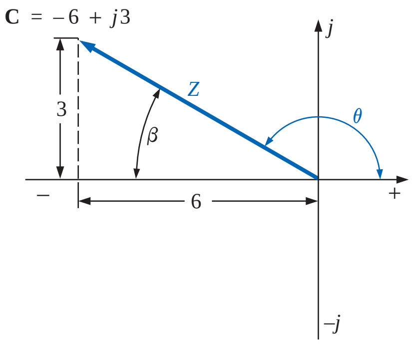


FIG. 14.48
Example 14.17.

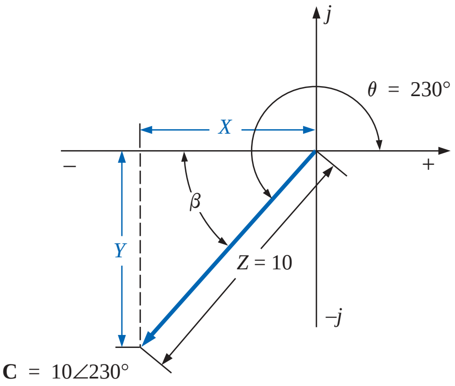


FIG. 14.49
Example 14.18.

EXAMPLE 14.18 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10 \angle 230° \qquad \text{(Fig. 14.49)}$$

Solution:

$$X = Z \cos \beta = 10 \cos(230° - 180°) = 10 \cos 50°$$
$$= (10)(0.6428) = 6.428$$
$$Y = Z \sin \beta = 10 \sin 50° = (10)(0.7660) = 7.660$$

and
$$\mathbf{C = -6.428 - \mathit{j}7.660}$$

14.10 MATHEMATICAL OPERATIONS WITH COMPLEX NUMBERS

Complex numbers lend themselves readily to the basic mathematical operations of addition, subtraction, multiplication, and division. A few basic rules and definitions must be understood before considering these operations.

Let us first examine the symbol j associated with imaginary numbers. By definition,

$$\boxed{j = \sqrt{-1}} \qquad \textbf{(14.27)}$$

Thus,
$$\boxed{j^2 = -1} \qquad \textbf{(14.28)}$$

and
$$j^3 = j^2 j = -1j = -j$$

with
$$j^4 = j^2 j^2 = (-1)(-1) = +1$$
$$j^5 = j$$

and so on. Further,

$$\frac{1}{j} = (1)\left(\frac{1}{j}\right) = \left(\frac{j}{j}\right)\left(\frac{1}{j}\right) = \frac{j}{j^2} = \frac{j}{-1}$$

and
$$\boxed{\frac{1}{j} = -j} \qquad \textbf{(14.29)}$$

Complex Conjugate

The **conjugate** or **complex conjugate** of a complex number can be found by simply changing the sign of the imaginary part in the rectangular form or by using the negative of the angle of the polar form. For example, the conjugate of

$$\mathbf{C} = 2 + j3$$

is
$$2 - j3$$

as shown in Fig. 14.50. The conjugate of

$$\mathbf{C} = 2 \angle 30°$$

is
$$2 \angle -30°$$

as shown in Fig. 14.51.

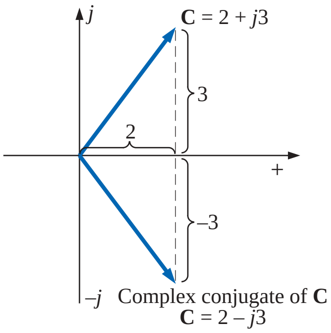


FIG. 14.50
Defining the complex conjugate of a complex number in rectangular form.

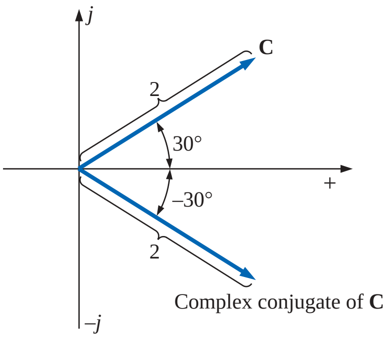


FIG. 14.51
Defining the complex conjugate of a complex number in polar form.

Reciprocal

The **reciprocal** of a complex number is 1 divided by the complex number. For example, the reciprocal of

$$\mathbf{C} = X + jY$$

is

$$\frac{1}{X + jY}$$

and of $Z \angle \theta$,

$$\frac{1}{Z \angle \theta}$$

We are now prepared to consider the four basic operations of *addition*, *subtraction*, *multiplication*, and *division* with complex numbers.

Addition

To add two or more complex numbers, simply add the real and imaginary parts separately. For example, if

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then

$$\boxed{\mathbf{C}_1 + \mathbf{C}_2 = (\pm X_1 \pm X_2) + j(\pm Y_1 \pm Y_2)} \qquad \textbf{(14.30)}$$

There is really no need to memorize the equation. Simply set one above the other and consider the real and imaginary parts separately, as shown in Example 14.19.

EXAMPLE 14.19

a. Add $\mathbf{C}_1 = 2 + j4$ and $\mathbf{C}_2 = 3 + j1$.
b. Add $\mathbf{C}_1 = 3 + j6$ and $\mathbf{C}_2 = -6 + j3$.

Solutions:

a. By Eq. (14.30),

$$\mathbf{C}_1 + \mathbf{C}_2 = (2 + 3) + j(4 + 1) = \mathbf{5 + j5}$$

Note Fig. 14.52. An alternative method is

$$\begin{array}{c} 2 + j4 \\ \underline{3 + j1} \\ \downarrow \quad \downarrow \\ \mathbf{5 + j5} \end{array}$$

b. By Eq. (14.30),

$$\mathbf{C}_1 + \mathbf{C}_2 = (3 - 6) + j(6 + 3) = \mathbf{-3 + j9}$$

Note Fig. 14.53. An alternative method is

$$\begin{array}{r} 3 + j6 \\ \underline{-6 + j3} \\ \downarrow \quad \downarrow \\ \mathbf{-3 + j9} \end{array}$$

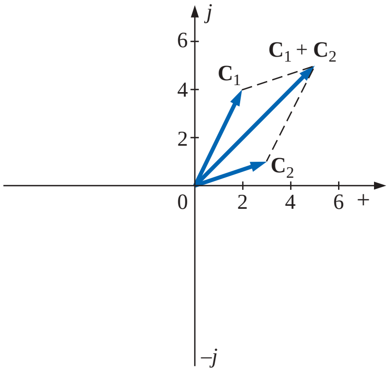


FIG. 14.52
Example 14.19(a).

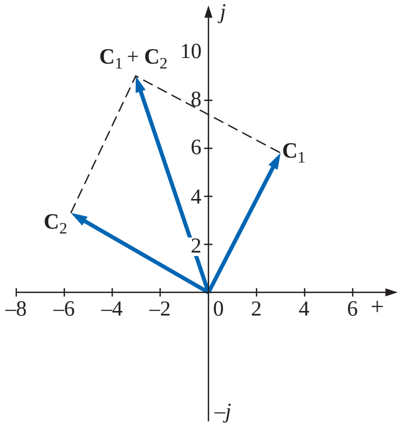


FIG. 14.53
Example 14.19(b).

Subtraction

In subtraction, the real and imaginary parts are again considered separately. For example, if

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then

$$\boxed{\mathbf{C}_1 - \mathbf{C}_2 = [\pm X_2 - (\pm X_2)] + j[\pm Y_1 - (\pm Y_2)]} \qquad \textbf{(14.31)}$$

Again, there is no need to memorize the equation if the alternative method of Example 14.20 is employed.

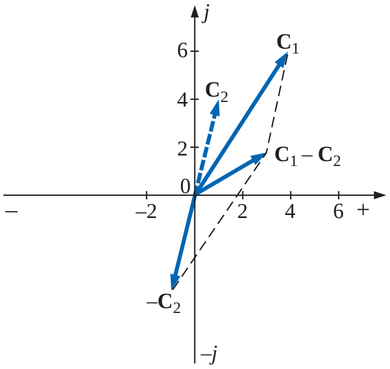


FIG. 14.54
Example 14.20(a).

EXAMPLE 14.20

a. Subtract $\mathbf{C}_2 = 1 + j4$ from $\mathbf{C}_1 = 4 + j6$.
b. Subtract $\mathbf{C}_2 = -2 + j5$ from $\mathbf{C}_1 = +3 + j3$.

Solutions:

a. By Eq. (14.31),

$$\mathbf{C}_1 - \mathbf{C}_2 = (4 - 1) + j(6 - 4) = \mathbf{3 + \mathit{j}2}$$

Note Fig. 14.54. An alternative method is

$$\begin{array}{c} 4 + j6 \\ \underline{-(1 + j4)} \\ \downarrow \quad \downarrow \\ \mathbf{3 + \mathit{j}2} \end{array}$$

b. By Eq. (14.31),

$$\mathbf{C}_1 - \mathbf{C}_2 = [3 - (-2)] + j(3 - 5) = \mathbf{5 - \mathit{j}2}$$

Note Fig. 14.55. An alternative method is

$$\begin{array}{c} 3 + j3 \\ \underline{-(-2 + j5)} \\ \downarrow \quad \downarrow \\ \mathbf{5 - \mathit{j}2} \end{array}$$

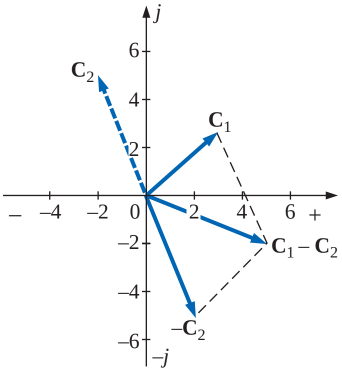


FIG. 14.55
Example 14.20(b).

Addition or subtraction cannot be performed in polar form unless the complex numbers have the same angle θ or unless they differ only by multiples of 180°.

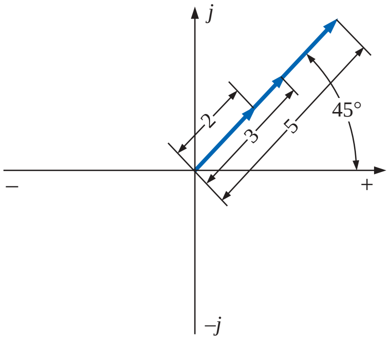


FIG. 14.56
Example 14.21(a).

EXAMPLE 14.21

a. $2 \angle 45° + 3 \angle 45° = \mathbf{5 \angle 45°}$

Note Fig. 14.56. Or

b. $2 \angle 0° - 4 \angle 180° = \mathbf{6 \angle 0°}$

Note Fig. 14.57.

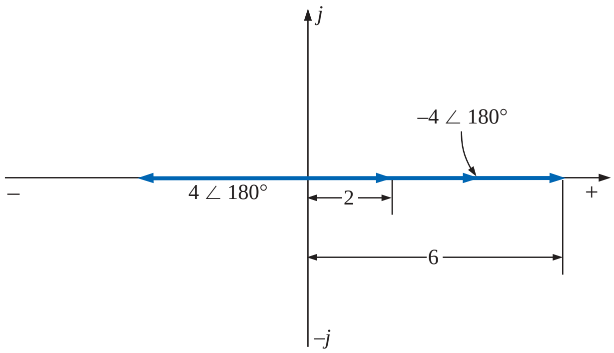


FIG. 14.57
Example 14.21(b).

Multiplication

To multiply two complex numbers in *rectangular* form, multiply the real and imaginary parts of one in turn by the real and imaginary parts of the other. For example, if

$$\mathbf{C}_1 = X_1 + jY_1 \quad \text{and} \quad \mathbf{C}_2 = X_2 + jY_2$$

then

$$\mathbf{C}_1 \cdot \mathbf{C}_2: \quad \begin{array}{r} X_1 + jY_1 \\ \underline{X_2 + jY_2} \\ X_1X_2 + jY_1X_2 \\ \underline{+ jX_1Y_2 + j^2Y_1Y_2} \\ X_1X_2 + j(X_1Y_1X_2 + X_1Y_2) + Y_1Y_2(-1) \end{array}$$

and

$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = (X_1X_2 - Y_1Y_2) + j(Y_1X_2 + X_1Y_2)} \qquad \textbf{(14.32)}$$

In Example 14.22(b), we obtain a solution without resorting to memorizing Eq. (14.32). Simply carry along the *j* factor when multiplying each part of one vector with the real and imaginary parts of the other.

EXAMPLE 14.22

a. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 2 + j3 \quad \text{and} \quad \mathbf{C}_2 = 5 + j10$$

b. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = -2 - j3 \quad \text{and} \quad \mathbf{C}_2 = +4 - j6$$

Solutions:

a. Using the format above, we have

$$\begin{aligned} \mathbf{C}_1 \cdot \mathbf{C}_2 &= [(2)(5) - (3)(10)] + j[(3)(5) + (2)(10)] \\ &= \mathbf{-20 + j35} \end{aligned}$$

b. Without using the format, we obtain

$$\begin{array}{l} -2 - j3 \\ \underline{+4 - j6 } \\ -8 - j12 \\ \underline{ + j12 + j^218} \\ -8 + j(-12 + 12) - 18 \end{array}$$

and

$$\mathbf{C}_1 \cdot \mathbf{C}_2 = \mathbf{-26 = 26 \angle 180°}$$

In *polar* form, the magnitudes are multiplied and the angles added algebraically. For example, for

$$\mathbf{C}_1 = Z_1 \angle \theta_1 \quad \text{and} \quad \mathbf{C}_2 = Z_2 \angle \theta_2$$

we write

$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = Z_1 Z_2 \; \underline{/\theta_1 + \theta_2}} \tag{14.33}$$

EXAMPLE 14.23

a. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 5 \angle 20^\circ \quad \text{and} \quad \mathbf{C}_2 = 10 \angle 30^\circ$$

b. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 2 \angle -40^\circ \quad \text{and} \quad \mathbf{C}_2 = 7 \angle +120^\circ$$

Solutions:

a. $\mathbf{C}_1 \cdot \mathbf{C}_2 = (5 \angle 20^\circ)(10 \angle 30^\circ) = (5)(10) \, /20^\circ + 30^\circ = \mathbf{50 \angle 50^\circ}$

b. $\mathbf{C}_1 \cdot \mathbf{C}_2 = (2 \angle -40^\circ)(7 \angle +120^\circ) = (2)(7) \, /-40^\circ + 120^\circ$
$= \mathbf{14 \angle +80^\circ}$

To multiply a complex number in rectangular form by a real number requires that both the real part and the imaginary part be multiplied by the real number. For example,

$$(10)(2 + j3) = 20 + j30$$

and $$50 \angle 0^\circ (0 + j6) = j300 = 300 \angle 90^\circ$$

Division

To divide two complex numbers in *rectangular* form, multiply the numerator and denominator by the conjugate of the denominator and the resulting real and imaginary parts collected. That is, if

$$\mathbf{C}_1 = X_1 + jY_1 \quad \text{and} \quad \mathbf{C}_2 = X_2 + jY_2$$

then
$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{(X_1 + jY_1)(X_2 - jY_2)}{(X_2 + jY_2)(X_2 - jY_2)}$$
$$= \frac{(X_1X_2 + Y_1Y_2) + j(X_2Y_1 - X_1Y_2)}{X_2^2 + Y_2^2}$$

and
$$\boxed{\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{X_1X_2 + Y_1Y_2}{X_2^2 + Y_2^2} + j\frac{X_2Y_1 - X_1Y_2}{X_2^2 + Y_2^2}} \tag{14.34}$$

The equation does not have to be memorized if the steps above used to obtain it are employed. That is, first multiply the numerator by the complex conjugate of the denominator and separate the real and imaginary terms. Then divide each term by the sum of each term of the denominator squared.

EXAMPLE 14.24

a. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 1 + j4$ and $\mathbf{C}_2 = 4 + j5$.

b. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = -4 - j8$ and $\mathbf{C}_2 = +6 - j1$.

Solutions:

a. By Eq. (14.34),

$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{(1)(4) + (4)(5)}{4^2 + 5^2} + j\,\frac{(4)(4) - (1)(5)}{4^2 + 5^2}$$

$$= \frac{24}{41} + \frac{j\,11}{41} \cong \mathbf{0.585 + \mathit{j}\,0.268}$$

b. Using an alternative method, we obtain

$$\begin{array}{l} -4 - j\,8 \\ \underline{+6 + j\,1} \\ -24 - j\,48 \\ \underline{- j\,4 - j^2 8} \\ -24 - j\,52 + 8 = -16 - j\,52 \end{array}$$

$$\begin{array}{l} +6 - j\,1 \\ \underline{+6 + j\,1} \\ 36 + j\,6 \\ \underline{- j\,6 - j^2 1} \\ 36 + 0 + 1 \quad = 37 \end{array}$$

and $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{-16}{37} - \frac{j\,52}{37} = \mathbf{-0.432 - \mathit{j}1.405}$$

To divide a complex number in rectangular form by a real number, both the real part and the imaginary part must be divided by the real number. For example,

$$\frac{8 + j\,10}{2} = 4 + j\,5$$

and $$\frac{6.8 - j\,0}{2} = 3.4 - j\,0 = 3.4\,\angle 0°$$

In *polar* form, division is accomplished by simply dividing the magnitude of the numerator by the magnitude of the denominator and subtracting the angle of the denominator from that of the numerator. That is, for

$$\mathbf{C}_1 = Z_1\,\angle\theta_1 \quad \text{and} \quad \mathbf{C}_2 = Z_2\,\angle\theta_2$$

we write

$$\boxed{\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{Z_1}{Z_2}\,\underline{/\theta_1 - \theta_2}} \qquad \textbf{(14.35)}$$

EXAMPLE 14.25

a. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 15\,\angle 10°$ and $\mathbf{C}_2 = 2\,\angle 7°$.

b. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 8\,\angle 120°$ and $\mathbf{C}_2 = 16\,\angle -50°$.

Solutions:

a. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{15\,\angle 10°}{2\,\angle 7°} = \frac{15}{2}\,\underline{/10° - 7°} = \mathbf{7.5\,\angle 3°}$$

b. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{8\angle 120^\circ}{16\angle -50^\circ} = \frac{8}{16}\,\underline{/120^\circ - (-50^\circ)} = \mathbf{0.5\angle 170^\circ}$$

We obtain the *reciprocal* in the rectangular form by multiplying the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{X + jY} = \left(\frac{1}{X + jY}\right)\left(\frac{X - jY}{X - jY}\right) = \frac{X - jY}{X^2 + Y^2}$$

and

$$\boxed{\frac{1}{X + jY} = \frac{X}{X^2 + Y^2} - j\,\frac{Y}{X^2 + Y^2}} \qquad \textbf{(14.36)}$$

In polar form, the reciprocal is

$$\boxed{\frac{1}{Z\angle\theta} = \frac{1}{Z}\angle -\theta} \qquad \textbf{(14.37)}$$

A concluding example using the four basic operations follows.

EXAMPLE 14.26 Perform the following operations, leaving the answer in polar or rectangular form:

a. $$\begin{aligned}\frac{(2 + j3) + (4 + j6)}{(7 + j7) - (3 - j3)} &= \frac{(2 + 4) + j(3 + 6)}{(7 - 3) + j(7 + 3)}\\ &= \frac{(6 + j9)(4 - j10)}{(4 + j10)(4 - j10)}\\ &= \frac{[(6)(4) + (9)(10)] + j[(4)(9) - (6)(10)]}{4^2 + 10^2}\\ &= \frac{114 - j24}{116} = \mathbf{0.983 - j0.207}\end{aligned}$$

b. $$\begin{aligned}\frac{(50\angle 30^\circ)(5 + j5)}{10\angle -20^\circ} &= \frac{(50\angle 30^\circ)(7.07\angle 45^\circ)}{10\angle -20^\circ} = \frac{353.5\angle 75^\circ}{10\angle -20^\circ}\\ &= 35.35\,\underline{/75^\circ - (-20^\circ)} = \mathbf{35.35\angle 95^\circ}\end{aligned}$$

c. $$\begin{aligned}\frac{(2\angle 20^\circ)^2(3 + j4)}{8 - j6} &= \frac{(2\angle 20^\circ)(2\angle 20^\circ)(5\angle 53.13^\circ)}{10\angle -36.87^\circ}\\ &= \frac{(4\angle 40^\circ)(5\angle 53.13^\circ)}{10\angle -36.87^\circ} = \frac{20\angle 93.13^\circ}{10\angle -36.87^\circ}\\ &= 2\,\underline{/93.13^\circ - (-36.87^\circ)} = \mathbf{2.0\angle 130^\circ}\end{aligned}$$

d. $$\begin{aligned}3\angle 27^\circ - 6\angle -40^\circ &= (2.673 + j1.362) - (4.596 - j3.857)\\ &= (2.673 - 4.596) + j(1.362 + 3.857)\\ &= \mathbf{-1.923 + j5.219}\end{aligned}$$

14.11 CALCULATOR AND COMPUTER METHODS WITH COMPLEX NUMBERS

The process of converting from one form to another or working through lengthy operations with complex numbers can be time-consuming and often frustrating if one lost minus sign or decimal point invalidates the solution. Fortunately, technologists of today have calculators and computer methods that make the process measurably easier with higher degrees of reliability and accuracy.

Calculators

The TI-86 calculator of Fig. 14.58 is only one of numerous calculators that can convert from one form to another and perform lengthy calculations with complex numbers in a concise, neat form. Not all of the details of using a specific calculator will be included here because each has its own format and sequence of steps. However, the basic operations with the TI-86 will be included primarily to demonstrate the ease with which the conversions can be made and the format for more complex operations.

For the TI-86 calculator, one must first call up the 2nd function CPLX from the keyboard, which results in a menu at the bottom of the display including conj, real, imag, abs, and angle. If we choose the key MORE, ► Rec and ► Pol will appear as options (for the conversion process). To convert from one form to another, simply enter the current form in brackets with a comma between components for the rectangular form and an angle symbol for the polar form. Follow this form with the operation to be performed, and press the ENTER key—the result will appear on the screen in the desired format.

FIG. 14.58

TI-86 scientific calculator. (Courtesy of Texas Instruments, Inc.)

EXAMPLE 14.27 This example is for demonstration purposes only. It is not expected that all readers will have a TI-86 calculator. The sole purpose of the example is to demonstrate the power of today's calculators.

Using the TI-86 calculator, perform the following conversions:

a. $3 - j4$ to polar form.

b. $0.006 \angle 20.6^\circ$ to rectangular form.

Solutions:

a. The TI-86 display for part (a) is the following:

```
(3, −4) ► Pol ENTER
(5.000E0∠−53.130E0)
```

CALC. 14.1

b. The TI-86 display for part (b) is the following:

```
(0.006∠20.6) ► Rec ENTER
(5.616E−3, 2.111E−3)
```

CALC. 14.2

EXAMPLE 14.28 Using the TI-86 calculator, perform the desired operations required in part (c) of Example 14.26, and compare solutions.

Solution: One must now be aware of the hierarchy of mathematical operations. In other words, in which sequence will the calculator perform the desired operations? In most cases, the sequence is the same as that used in longhand calculations, although one must become adept at setting up the parentheses to ensure the correct order of operations. For this example, the TI-86 display is the following:

$((2\angle 20)^2 * (3,4))/(8,-6)$ ▶ Pol ENTER

$(2.000E0\angle 130.000E0)$

CALC. 14.3

which is a perfect match with the earlier solution.

Mathcad

The Mathcad format for complex numbers will now be introduced in preparation for the chapters to follow. We will continue to use *j* when we define a complex number in rectangular form even though the Mathcad result will always appear with the letter *i*. You can change this by going to the **Format** menu, but for this presentation we decided to use the default operators as much as possible.

When entering *j* to define the imaginary component of a complex number, be sure to enter it as 1*j*; but do not put a multiplication operator between the 1 and the *j*. Just type 1 and then *j*. In addition, place the *j* after the constant rather than before as in the text material.

When Mathcad operates on an angle, it will assume that the angle is in radians and not degrees. Further, all results will appear in radians rather than degrees.

The first operation to be developed is the conversion from rectangular to polar form. In Fig. 14.59 the rectangular number $4 + j\,3$ is being converted to polar form using Mathcad. First X and Y are defined using the colon operator. Next the equation for the magnitude of the polar form is written in terms of the two variables just defined. The magnitude of the polar form is then revealed by writing the variable again and using the equal sign. It will take some practice, but be careful when writing the equation for Z in the sense that you pay particular attention to the location of the bracket before performing the next operation. The resulting magnitude of 5 is as expected.

For the angle, the sequence **View-Toolbars-Greek** is first applied to obtain the **Greek** toolbar appearing in Fig. 14.59. It can be moved to any location by simply clicking on the blue at the top of the toolbar and dragging it to the preferred location. Then 0 is selected from the toolbar as the variable to be defined. The $\tan^{-1}\theta$ is obtained through the sequence **Insert-*f(x)*-Insert Function** dialog box-**trigonometric-atan-OK** in which Y/X is inserted. Then bring the controlling bracket to the outside of the entire expression, and multiply by the ratio of $180/\pi$ with π selected from the **Calculator** toolbar (available from the same sequence used to obtain the **Greek** toolbar). The multiplication by the last factor of the equation will ensure that the angle is in degrees. Selecting θ again followed by an equal sign will result in the correct angle of 36.87° as shown in Fig. 14.59.

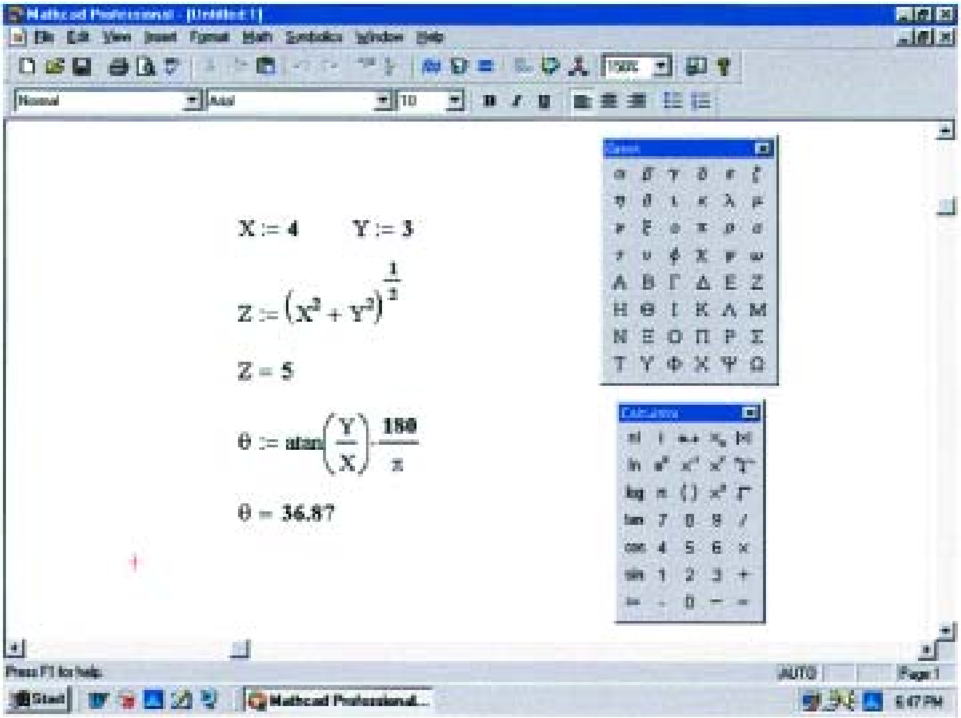
We will now look at two forms for the polar form of a complex number. The first is defined by the basic equations introduced in this chapter, while the second uses a special format. For all the Mathcad analyses to be provided in this text, the latter format will be employed. First

German-American
(Breslau, Germany;
Yonkers and
Schenectady,
NY, USA)
(1865–1923)
Mathematician,
Scientist,
Engineer, Inventor,
Professor of
Electrical
Engineering and
Electrophysics,
Union College
Department Head,
General Electric Co.

Courtesy of the
Hall of History Foundation,
Schenectady, New York

Although the holder of some 200 patents and recognized worldwide for his contributions to the study of hysteresis losses and electrical transients, Charles Proteus Steinmetz is best recognized for his contribution to the study of ac networks. His “Symbolic Method of Alternating-current Calculations” provided an approach to the analysis of ac networks that removed a great deal of the confusion and frustration experienced by engineers of that day as they made the transition from dc to ac systems. His approach (from which the phasor notation of this text is premised) permitted a direct analysis of ac systems using many of the theorems and methods of analysis developed for dc systems. In 1897 he authored the epic work *Theory and Calculation of Alternating Current Phenomena*, which became the “bible” for practicing engineers. Dr. Steinmetz was fondly referred to as “The Doctor” at General Electric Company where he worked for some 30 years in a number of important capacities. His recognition as a “multigifted genius” is supported by the fact that he maintained active friendships with such individuals as Albert Einstein, Guglielmo Marconi (radio), and Thomas A. Edison, to name just a few. He was President of the American Institute of Electrical Engineers (AIEE) and the National Association of Corporation Schools and actively supported his local community (Schenectady) as president of the Board of Education and the Commission on Parks and City Planning.

CHARLES PROTEUS STEINMETZ



$X := 4 \qquad Y := 3$

$Z := \left(X^2 + Y^2\right)^{\frac{1}{2}}$

$Z = 5$

$\theta := \operatorname{atan}\left(\frac{Y}{X}\right)\cdot\frac{180}{\pi}$

$\theta = 36.87$

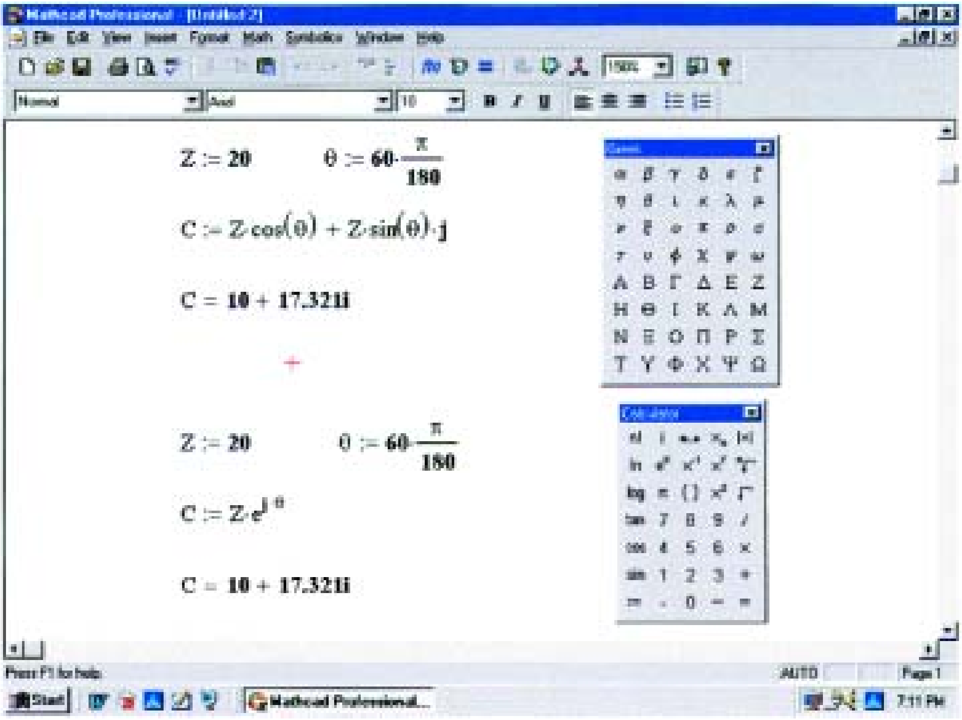
FIG. 14.59

Using Mathcad to convert from rectangular to polar form.

the magnitude of the polar form is defined followed by the conversion of the angle of 60° to radians by multiplying by the factor $\pi/180$ as shown in Fig. 14.60. In this example the resulting angular measure is $\pi/3$ radians. Next the rectangular format is defined by a real part $X = Z \cos\theta$ and by an imaginary part $Y = Z \sin\theta$. Both the cos and the sin are obtained by the sequence **Insert-*f(x)*-trigonometric-cos(**or **sin)-OK.** Note the multiplication by *j* which was actually entered as 1*j*. Entering *C* again followed by an equal sign will result in the correct conversion shown in Fig. 14.60.

The next format is based on the mathematical relationship that $e^{j\theta} = \cos\theta + j\sin\theta$. Both *Z* and θ are as defined above, but now the complex number is written as shown in Fig. 14.60 using the notation just introduced. Note that both *Z* and θ are part of this defining form. The e^x is obtained directly from the **Calculator** toolbar. Remember to enter the *j* as 1*j* without a multiplication sign between the 1 and the *j*. However, there is a multiplication operator placed between the *j* and θ. When entered again followed by an equal sign, the rectangular form appears to match the above results. As mentioned above, it is this latter format that will be used throughout the text due to its cleaner form and more direct entering path.

The last example using Mathcad will be a confirmation of the results of Example 14.26(b) as shown in Fig. 14.61. The three complex numbers are first defined as shown. Then the equation for the desired result



$Z := 20 \qquad \theta := 60\cdot\frac{\pi}{180}$

$C := Z\cdot\cos(\theta) + Z\cdot\sin(\theta)\cdot j$

$C = 10 + 17.321i$

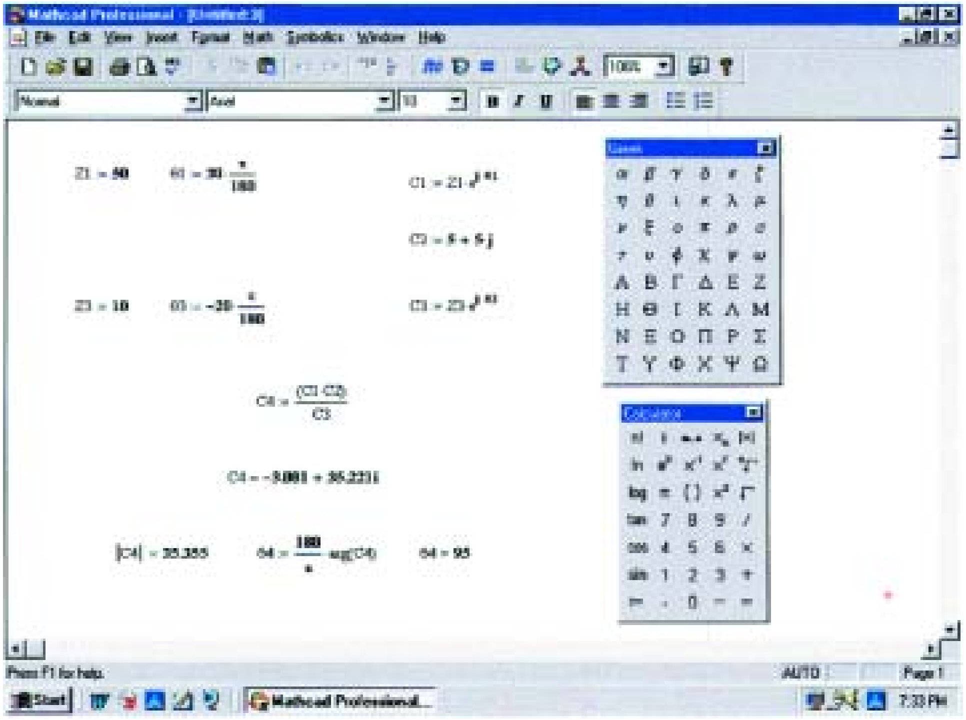
$Z := 20 \qquad \theta := 60\cdot\frac{\pi}{180}$

$C := Z\cdot e^{j\cdot\theta}$

$C = 10 + 17.321i$

FIG. 14.60

Using Mathcad to convert from polar to rectangular form.



$Z1 := 50 \qquad \theta1 := 30\cdot\frac{\pi}{180} \qquad C1 := Z1\cdot e^{j\cdot\theta1}$

$C2 := 5 + 5\cdot j$

$Z3 := 10 \qquad \theta3 := -20\cdot\frac{\pi}{180} \qquad C3 := Z3\cdot e^{j\cdot\theta3}$

$C4 := \frac{(C1\cdot C2)}{C3}$

$C4 = -3.081 + 35.221i$

$|C4| = 35.355 \qquad \theta4 := \frac{180}{\pi}\cdot\arg(C4) \qquad \theta4 = 95$

FIG. 14.61

Using Mathcad to confirm the results of Example 14.26(b).

is entered using C_4, and finally the results are called for. Note the relative simplicity of the equation for C_4 now that all the other variables have been defined. As shown, however, the immediate result is in the rectangular form using the magnitude feature from the calculator and the **arg** function from **Insert-*f(x)*-Complex Numbers-arg.** There will be a number of other examples in the chapters to follow on the use of Mathcad with complex numbers.

14.12 PHASORS

As noted earlier in this chapter, the addition of sinusoidal voltages and currents will frequently be required in the analysis of ac circuits. One lengthy but valid method of performing this operation is to place both sinusoidal waveforms on the same set of axes and add algebraically the magnitudes of each at every point along the abscissa, as shown for $c = a + b$ in Fig. 14.62. This, however, can be a long and tedious process with limited accuracy. A shorter method uses the rotating radius vector first appearing in Fig. 13.16. This *radius vector,* having a *constant magnitude* (length) with *one end fixed at the origin,* is called a **phasor** when applied to electric circuits. During its rotational development of the sine wave, the phasor will, at the instant $t = 0$, have the positions shown in Fig. 14.63(a) for each waveform in Fig. 14.63(b).

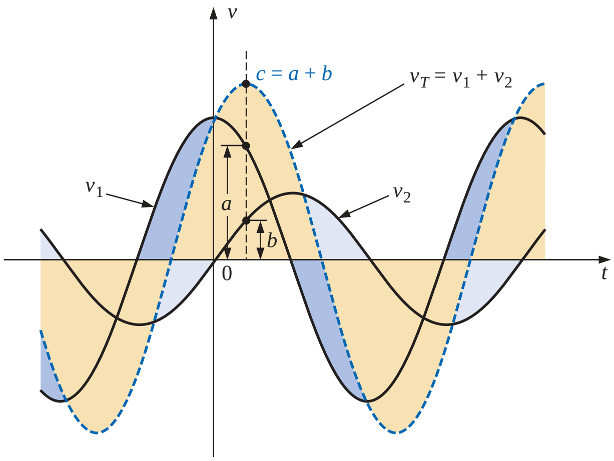
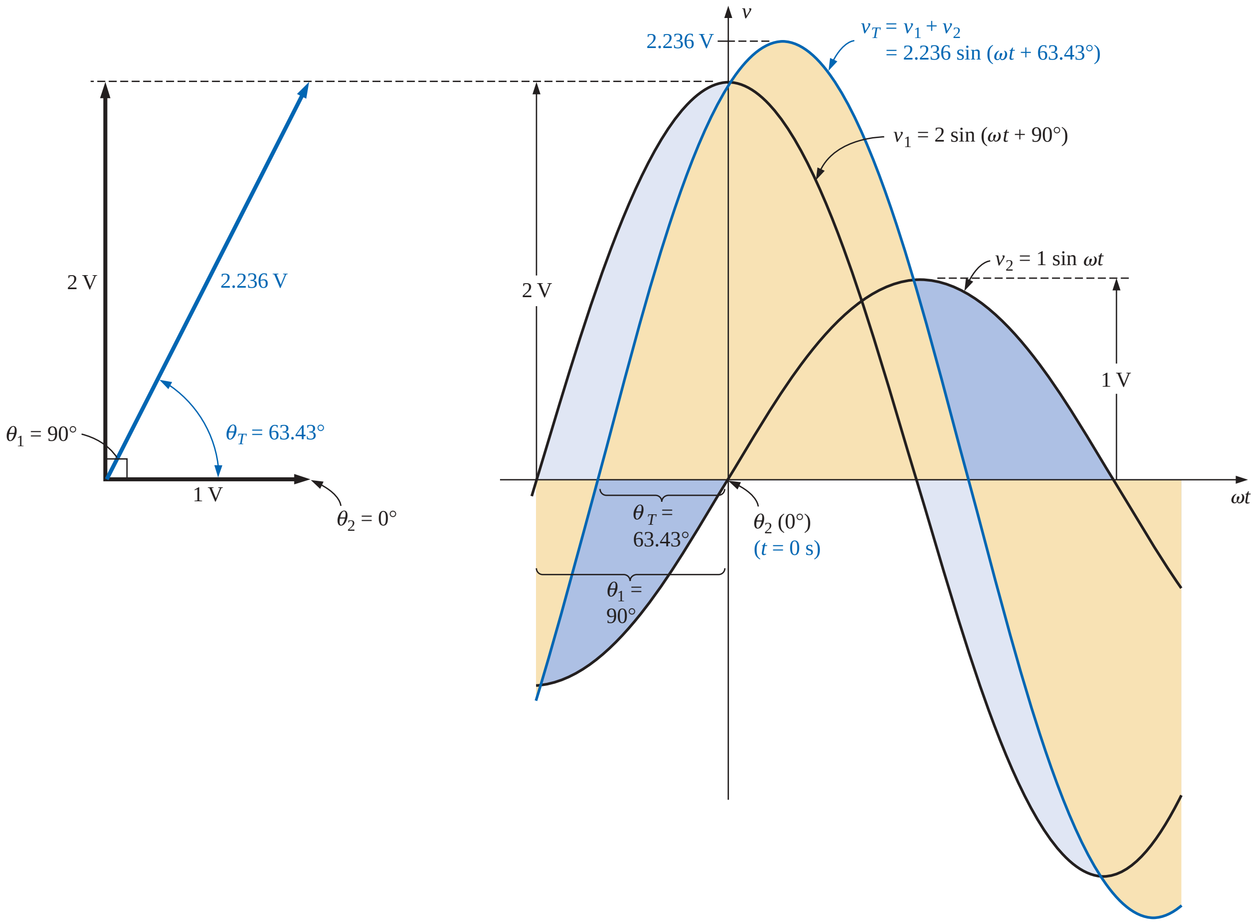


FIG. 14.62
Adding two sinusoidal waveforms on a point-by-point basis.

Note in Fig. 14.63(b) that v_2 passes through the horizontal axis at $t = 0$ s, requiring that the radius vector in Fig. 14.63(a) be on the horizontal axis to ensure a vertical projection of zero volts at $t = 0$ s. Its length in Fig. 14.63(a) is equal to the peak value of the sinusoid as required by the radius vector of Fig. 13.16. The other sinusoid has passed through 90° of its rotation by the time $t = 0$ s is reached and



(a)

(b)

FIG. 14.63

(a) The phasor representation of the sinusoidal waveforms of Fig. 14.63(b); (b) finding the sum of two sinusoidal waveforms of v_1 and v_2.

therefore has its maximum vertical projection as shown in Fig. 14.63(a). Since the vertical projection is a maximum, the peak value of the sinusoid that it will generate is also attained at $t = 0$ s, as shown in Fig. 14.63(b). Note also that $v_T = v_1$ at $t = 0$ s since $v_2 = 0$ V at this instant.

It can be shown [see Fig. 14.63(a)] using the vector algebra described in Section 14.10 that

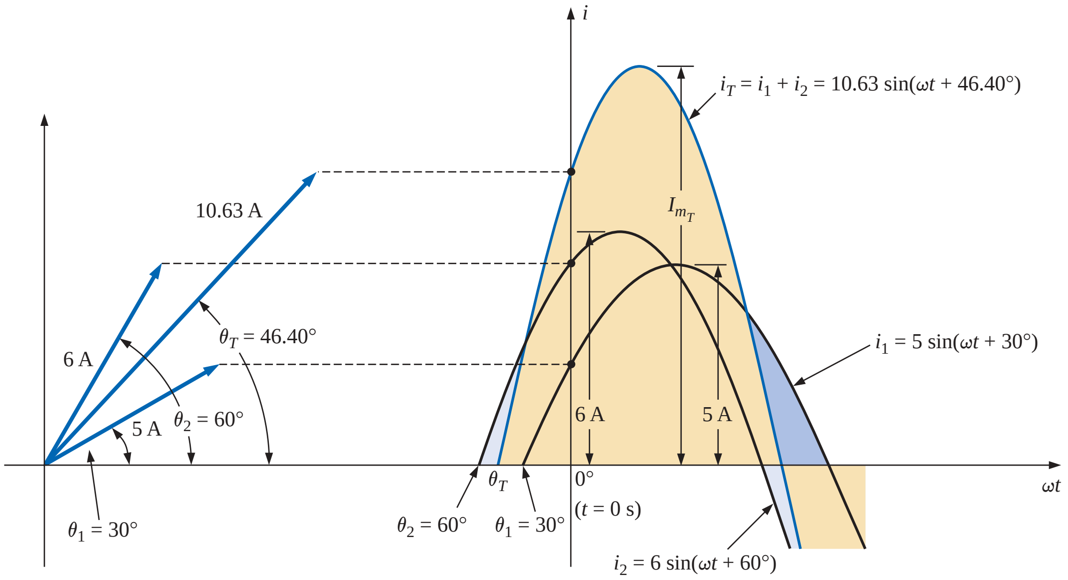
$$1 \text{ V} \angle 0° + 2 \text{ V} \angle 90° = 2.236 \text{ V} \angle 63.43°$$

In other words, if we convert v_1 and v_2 to the phasor form using

$$v = V_m \sin(\omega t \pm \theta) \Rightarrow V_m \angle \pm\theta$$

and add them using complex number algebra, we can find the phasor form for v_T with very little difficulty. It can then be converted to the time domain and plotted on the same set of axes, as shown in Fig. 14.63(b). Figure 14.63(a), showing the magnitudes and relative positions of the various phasors, is called a **phasor diagram.** It is actually a “snapshot” of the rotating radius vectors at $t = 0$ s.

In the future, therefore, if the addition of two sinusoids is required, they should first be converted to the phasor domain and the sum found using complex algebra. The result can then be converted to the time domain.



(a) (b)

FIG. 14.64

Adding two sinusoidal currents with phase angles other than 90°.

The case of two sinusoidal functions having phase angles different from 0° and 90° appears in Fig. 14.64. Note again that the vertical height of the functions in Fig. 14.64(b) at $t = 0$ s is determined by the rotational positions of the radius vectors in Fig. 14.64(a).

Since the rms, rather than the peak, values are used almost exclusively in the analysis of ac circuits, the phasor will now be redefined for the purposes of practicality and uniformity as having a magnitude equal to the *rms value* of the sine wave it represents. The angle associated with the phasor will remain as previously described—the phase angle.

In general, for all of the analyses to follow, the phasor form of a sinusoidal voltage or current will be

$$\mathbf{V} = V \angle \theta \quad \text{and} \quad \mathbf{I} = I \angle \theta$$

where V and I are rms values and θ is the phase angle. It should be pointed out that in phasor notation, the sine wave is always the reference, and the frequency is not represented.

Phasor algebra for sinusoidal quantities is applicable only for waveforms having the same frequency.

EXAMPLE 14.29 Convert the following from the time to the phasor domain:

Time Domain	Phasor Domain
a. $\sqrt{2}(50) \sin \omega t$	$\mathbf{50 \angle 0°}$
b. $69.6 \sin(\omega t + 72°)$	$(0.707)(69.6) \angle 72° = \mathbf{49.21 \angle 72°}$
c. $45 \cos \omega t$	$(0.707)(45) \angle 90° = \mathbf{31.82 \angle 90°}$

EXAMPLE 14.30 Write the sinusoidal expression for the following phasors if the frequency is 60 Hz:

Phasor Domain	Time Domain
a. $\mathbf{I} = 10\ \angle 30^\circ$	$i = \sqrt{2}(10)\sin(2\pi 60t + 30^\circ)$ and $i = \mathbf{14.14\ sin(377}\boldsymbol{t}\mathbf{ + 30^\circ)}$
b. $\mathbf{V} = 115\ \angle -70^\circ$	$v = \sqrt{2}(115)\sin(377t - 70^\circ)$ and $v = \mathbf{162.6\ sin(377}\boldsymbol{t}\mathbf{ - 70^\circ)}$

EXAMPLE 14.31 Find the input voltage of the circuit of Fig. 14.65 if

$$\left.\begin{aligned} v_a &= 50\sin(377t + 30^\circ) \\ v_b &= 30\sin(377t + 60^\circ) \end{aligned}\right\} f = 60\text{ Hz}$$

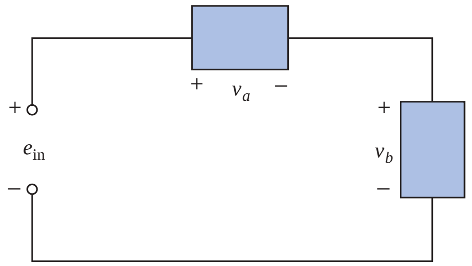


FIG. 14.65
Example 14.31.

Solution: Applying Kirchhoff's voltage law, we have

$$e_{\text{in}} = v_a + v_b$$

Converting from the time to the phasor domain yields

$$v_a = 50\sin(377t + 30^\circ) \Rightarrow \mathbf{V}_a = 35.35\text{ V}\ \angle 30^\circ$$
$$v_b = 30\sin(377t + 60^\circ) \Rightarrow \mathbf{V}_b = 21.21\text{ V}\ \angle 60^\circ$$

Converting from polar to rectangular form for addition yields

$$\mathbf{V}_a = 35.35\text{ V}\ \angle 30^\circ = 30.61\text{ V} + j\,17.68\text{ V}$$
$$\mathbf{V}_b = 21.21\text{ V}\ \angle 60^\circ = 10.61\text{ V} + j\,18.37\text{ V}$$

Then

$$\begin{aligned}\mathbf{E}_{\text{in}} = \mathbf{V}_a + \mathbf{V}_b &= (30.61\text{ V} + j\,17.68\text{ V}) + (10.61\text{ V} + j\,18.37\text{ V}) \\ &= 41.22\text{ V} + j\,36.05\text{ V}\end{aligned}$$

Converting from rectangular to polar form, we have

$$\mathbf{E}_{\text{in}} = 41.22\text{ V} + j\,36.05\text{ V} = 54.76\text{ V}\ \angle 41.17^\circ$$

Converting from the phasor to the time domain, we obtain

$$\mathbf{E}_{\text{in}} = 54.76\text{ V}\ \angle 41.17^\circ \Rightarrow e_{\text{in}} = \sqrt{2}(54.76)\sin(377t + 41.17^\circ)$$

and $$e_{\text{in}} = \mathbf{77.43\ sin(377}\boldsymbol{t}\mathbf{ + 41.17^\circ)}$$

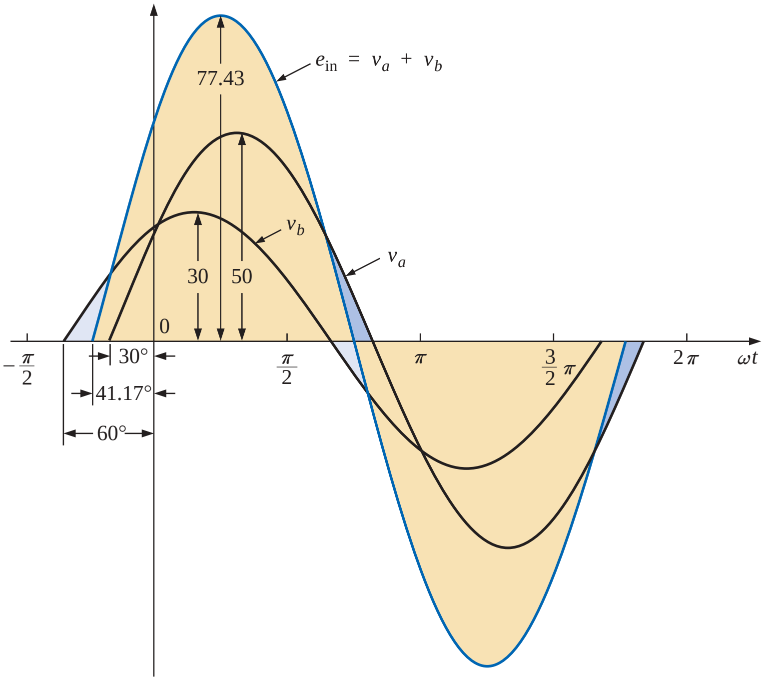


FIG. 14.66
Solution to Example 14.31.

A plot of the three waveforms is shown in Fig. 14.66. Note that at each instant of time, the sum of the two waveforms does in fact add up to e_{in}. At $t = 0$ ($\omega t = 0$), e_{in} is the sum of the two positive values, while at a value of ωt, almost midway between $\pi/2$ and π, the sum of the positive value of v_a and the negative value of v_b results in $e_{in} = 0$.

EXAMPLE 14.32 Determine the current i_2 for the network of Fig. 14.67.

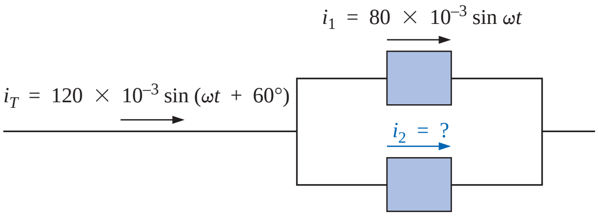


FIG. 14.67
Example 14.32.

Solution: Applying Kirchhoff's current law, we obtain

$$i_T = i_1 + i_2 \quad \text{or} \quad i_2 = i_T - i_1$$

Converting from the time to the phasor domain yields

$$i_T = 120 \times 10^{-3} \sin(\omega t + 60°) \Rightarrow 84.84 \text{ mA } \angle 60°$$
$$i_1 = 80 \times 10^{-3} \sin \omega t \Rightarrow 56.56 \text{ mA } \angle 0°$$

Converting from polar to rectangular form for subtraction yields

$$\mathbf{I}_T = 84.84\text{ mA }\angle 60^\circ = 42.42\text{ mA} + j\,73.47\text{ mA}$$
$$\mathbf{I}_1 = 56.56\text{ mA }\angle 0^\circ = 56.56\text{ mA} + j\,0$$

Then

$$\begin{aligned}\mathbf{I}_2 &= \mathbf{I}_T - \mathbf{I}_1\\ &= (42.42\text{ mA} + j\,73.47\text{ mA}) - (56.56\text{ mA} + j\,0)\end{aligned}$$

and $$\mathbf{I}_2 = -14.14\text{ mA} + j\,73.47\text{ mA}$$

Converting from rectangular to polar form, we have

$$\mathbf{I}_2 = 74.82\text{ mA }\angle 100.89^\circ$$

Converting from the phasor to the time domain, we have

$$\begin{aligned}\mathbf{I}_2 &= 74.82\text{ mA }\angle 100.89^\circ \Rightarrow\\ i_2 &= \sqrt{2}(74.82 \times 10^{-3})\sin(\omega t + 100.89^\circ)\end{aligned}$$

and $$i_2 = \mathbf{105.8 \times 10^{-3} \sin(\omega \mathit{t} + 100.89^\circ)}$$

A plot of the three waveforms appears in Fig. 14.68. The waveforms clearly indicate that $i_T = i_1 + i_2$.

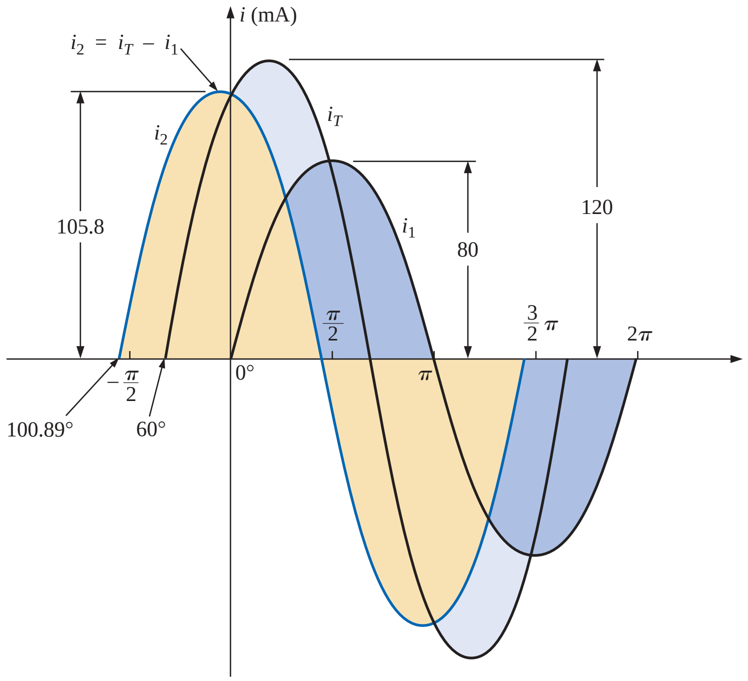
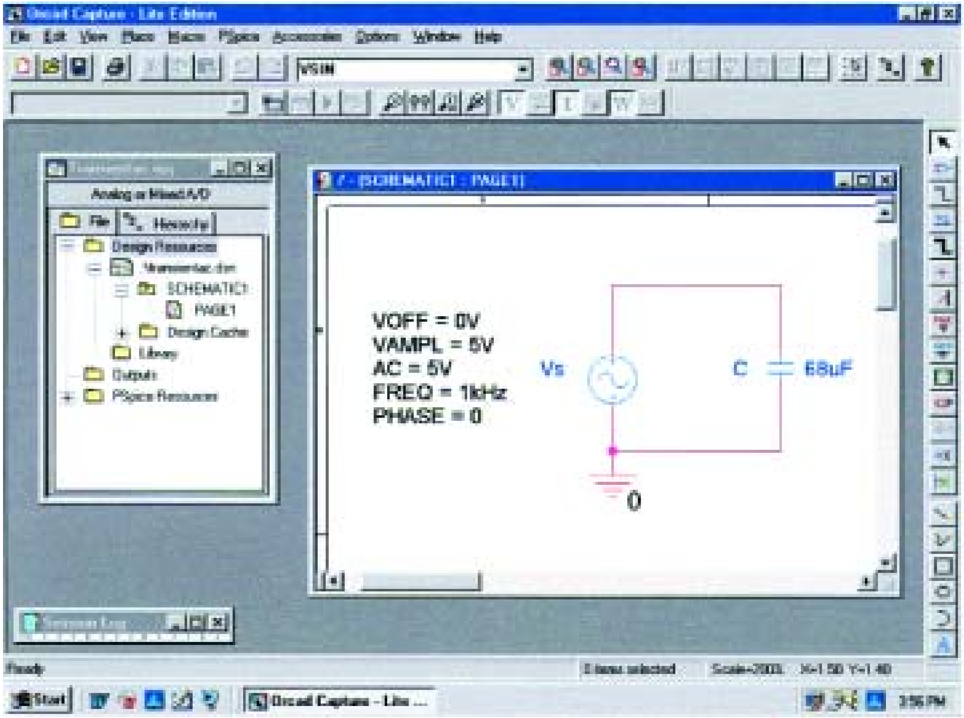


FIG. 14.68
Solution to Example 14.32.

14.13 COMPUTER ANALYSIS

PSpice

Capacitors and the ac Response The simplest of ac capacitive circuits will now be analyzed to introduce the process of setting up an ac source and running an ac transient simulation. The ac source of Fig.



VOFF = 0V
VAMPL = 5V
AC = 5V
FREQ = 1kHz
PHASE = 0

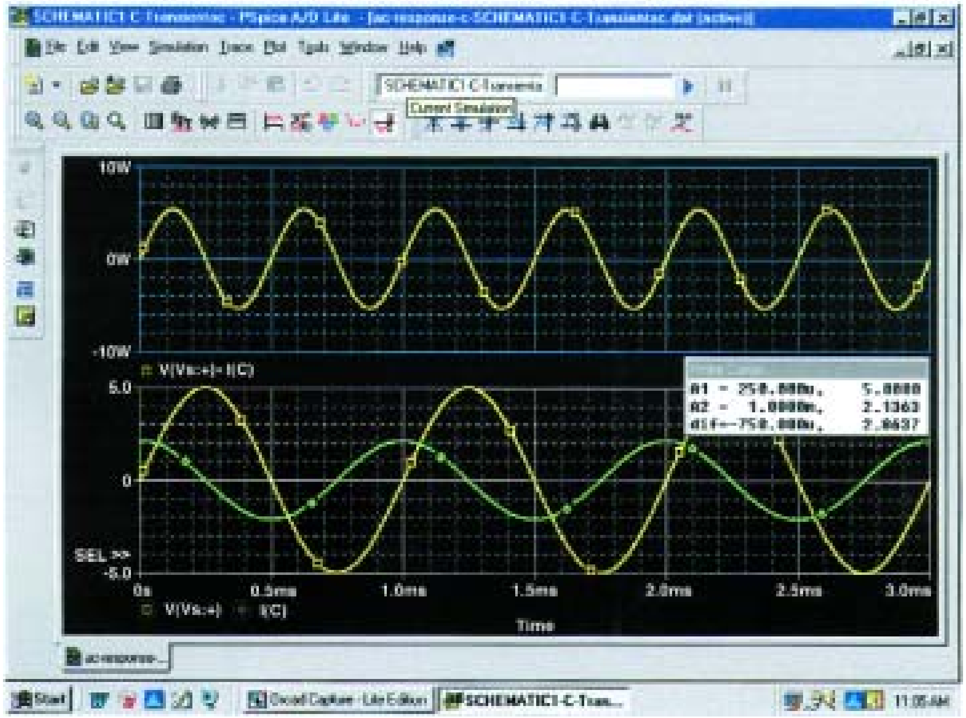
FIG. 14.69

Using PSpice to analyze the response of a capacitor to a sinusoidal ac signal.

14.69 is obtained through **Place part** key-**SOURCE-VSIN-OK.** The name or value of any parameter can be changed by simply double-clicking on the parameter on the display or by double-clicking on the source symbol to get the **Property Editor** dialog box. Within the dialog box the values appearing in Fig. 14.69 were set, and under **Display, Name and Value** were selected. After you have selected **Apply** and exited the dialog box, the parameters will appear as shown in the figure.

The simulation process is initiated by selecting the **New Simulation Profile** and under **New Simulation** entering **Transientac** for the **Name** followed by **Create.** In the **Simulation Settings** dialog box, **Analysis** is selected and **Time Domain(Transient)** is chosen under **Analysis type.** The **Run to time** will be set at 3 ms to permit a display of three cycles of the sinusoidal waveforms ($T = 1/f = 1/1000$ Hz $= 1$ ms). The **Start saving data after** will be left at 0 s, and the **Maximum step size** will be 3 ms/1000 $= 3$ μs. Clicking **OK** and then selecting the **Run PSpice** icon will result in a plot having a horizontal axis that extends from 0 to 3 ms.

Now we have to tell the computer which waveforms we are interested in. First, we should take a look at the applied ac source by selecting **Trace-Add Trace-V(Vs:+)** followed by **OK.** The result is the sweeping ac voltage in the botttom region of the screen of Fig. 14.70. Note that it has a peak value of 5 V, and three cycles appear in the 3-ms time frame. The current for the capacitor can be added by selecting **Trace-Add Trace** and choosing **I(C)** followed by **OK.** The resulting waveform for **I(C)** appears at a 90° phase shift from the applied voltage, with the current leading the voltage (the current has already peaked



```
A1 = 250.000u,     5.0000
A2 =   1.0000m,    2.1363
dif=-750.000u,     2.8637
```

FIG. 14.70

A plot of the voltage, current, and power for the capacitor of Fig. 14.69.

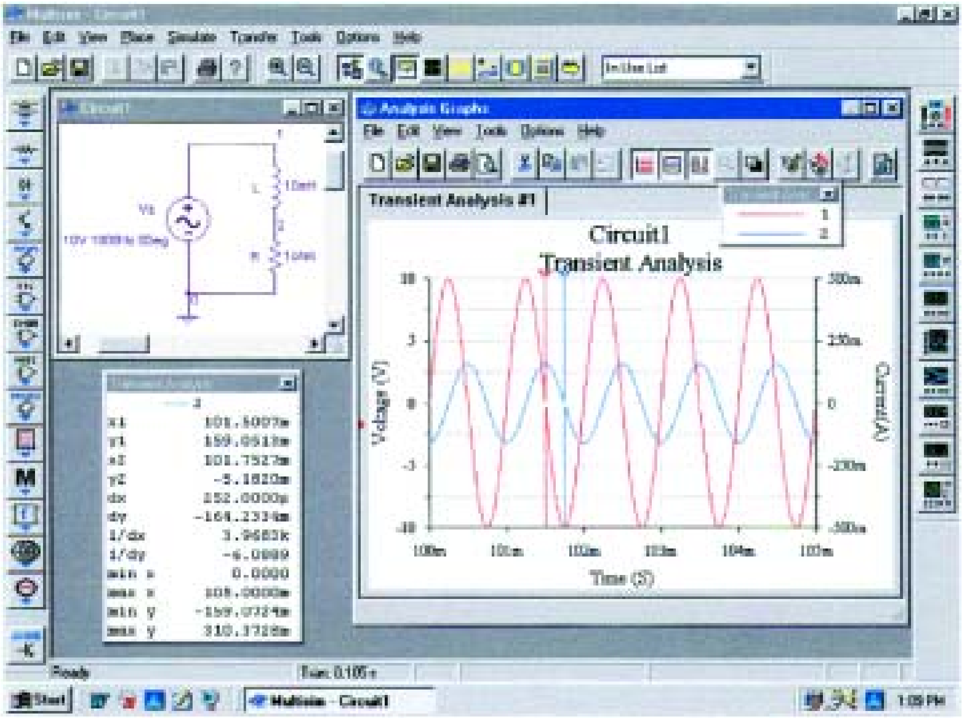
as the voltage crosses the 0-V axis). Since the peak value of each plot is in the same magnitude range, the 5 appearing on the vertical scale can be used for both. A theoretical analysis would result in $X_C = 2.34\ \Omega$, and the peak value of $I_C = E/X_C = 5\text{ V}/2.34\ \Omega = 2.136\text{ A}$, as shown in Fig. 14.70.

For interest sake, and a little bit of practice, let us obtain the curve for the power delivered to the capacitor over the same time period. First select **Plot-Add Plot to Window-Trace-Add Trace** to obtain the **Add Traces** dialog box. Then chose **V(Vs:+)**, follow it with a * for multiplication, and finish by selecting **I(C).** The result is the expression **V(Vs:+)*I(C)** of the power format: $p = vi$. Click **OK,** and the power plot at the top of Fig 14.70 will appear. Note that over the full three cycles, the area above the axis equals the area below—there is no net transfer of power over the 3-ms period. Note also that the power curve is sinusoidal (which is quite interesting) with a frequency twice that of the applied signal. Using the cursor control, we can determine that the maximum power (peak value of the sinusoidal waveform) is 5.34 W. The cursors, in fact, have been added to the lower curves to show the peak value of the applied sinusoid and the resulting current.

After selecting the **Toggle cursor** icon, left-click the mouse to surround the **V(Vs:+)** at the bottom of the plot with a dashed line to show that the cursor is providing the levels of that quantity. When placed at $\frac{1}{4}$ of the total period of 250 μs **(A1),** the peak value is exactly 5 V as shown in the **Probe Cursor** dialog box. Placing the cursor over the symbol next to **I(C)** at the bottom of the plot and right-clicking the mouse will assign the right cursor to the current. Placing it at exactly 1 ms **(A2)** will result in a peak value of 2.136 A to match the solution above. To further distinguish between the voltage and current waveforms, the color and the width of the lines of the traces were changed. Place the cursor right on the plot line and perform a right click. Then the **Properties** option appears. When **Properties** is selected, a **Trace Properties** dialog box will appear in which the yellow color can be selected and the width widened to improve the visibility on the black background. Note that yellow was chosen for **Vs** and green for **I(C)**. Note also that the axis and the grid have been changed to a more visible color using the same procedure.

Electronics Workbench

Since PSpice reviewed the response of a capacitive element to an ac voltage, Electronics Workbench will repeat the analysis for an inductive element. The ac voltage source was derived from the **Sources** parts bin as described in Chapter 13 with the values appearing in Fig. 14.71 set in the **AC Voltage** dialog box. Since the transient response of Electronics Workbench is limited to a plot of voltage versus time, a plot of the current of the circuit will require the addition of a resistor of 1 Ω in series



```
         2
x1       101.5007m
y1       159.0513m
x2       101.7527m
y2       -5.1820m
dx       252.0000µ
dy       -164.2334m
1/dx     3.9683k
1/dy     -6.0889
min x    0.0000
max x    105.0000m
min y    -159.0724m
max y    310.3728m
```

FIG. 14.71

Using Electronics Workbench to review the response of an inductive element to a sinusoidal ac signal.

with the inductive element. The magnitude of the current through the resistor and, of course, the series inductor will then be determined by

$$| i_R | = \left| \frac{v_R}{R} \right| = \left| \frac{v_R}{1\Omega} \right| = | v_R | = | i_L |$$

revealing that the current will have the same peak value as the voltage across the resistor due to the division by 1. When viewed on the graph, it can simply be considered a plot of the current. In actuality, all inductors require a series resistance, so the 1-Ω resistor serves an important dual purpose. The 1-Ω resistance is also so small compared to the reactance of the coil at the 1-kHz frequency that its effect on the total impedance or voltage across the coil can be ignored.

Once the circuit has been constructed, the sequence **Simulate-Analyses-Transient Analysis** will result in a **Transient Analysis** dialog box in which the **Start time** is set at 0 s and the **End time** at 105 ms. The 105 ms was set as the **End time** to give the network 100 ms to settle down in its steady-state mode and 5 ms for five cycles in the output display. The **Minimum number of time points** was set at 10,000 to ensure a good display for the rapidly changing waveforms.

Next the **Output variables** heading was chosen within the dialog box, and nodes **1** and **2** were moved from the **Variables in Circuit** to **Selected variables** for analysis using the **Plot during simulation** key pad. Choosing **Simulate** will then result in a waveform that extends from 0 s to 105 ms. Even though we plan to save only the response that occurs after 100 ms, the computer is unaware of our interest, and it plots the response for the entire period, This is corrected by selecting the **Properties** key pad in the toolbar at the top of the graph (it looks like a tag and pencil) to obtain the **Graph Properties** dialog box. Selecting **Bottom Axis** will permit setting the **Range** from a **Minimum of 0.100s=100ms** to a **Maximum of 0.105s=105ms.** Click **OK**, and the time period of Fig. 14.71 will be displayed. The grid structure is added by selecting the **Show/Hide Grid** key pad, and the color associated with each nodal voltage will be displayed if we choose the **Show/Hide Legend** key next to it.

The scale for the plot of i_L can be improved by first going to **Traces** and setting the **Trace** to the number **2** representing the voltage across the 1-Ω resistor. When **2** is selected, the **Color** displayed will automaticaly change to blue. In the **Y Range**, select **Right Axis** followed by **OK.** Then select the **Right Axis** heading, and enter **Current(A)** for the **Label**, enable **Axis**, change the **Pen Size** to 1, and change the **Range** from −500 mA to +500 mA. Finally, set the **Total Ticks** at 8 with **Minor Ticks** at 2 to match the **Left Axis**, and leave the box with an **OK.** The plot of Fig. 14.71 will result. Take immediate note of the new axis on the right and the **Current(A)** label. We can now see that the current has a peak of about 160 mA. For more detail on the peak values, simply click on the **Show/Hide Cursors** key pad on the top toolbar. A **Transient Analysis** dialog box will appear with a **1** and a red line to indicate that it is working on the full source voltage at node **1.** To switch to the current curve (the blue curve), simply bring the cursor to any point on the blue curve and perform a left click. A blue line and the number **2** will appear at the heading of the **Transient Analysis** dialog box. Clicking on the **1** in the small inverted arrow at the top will allow you to drag the vertical red line to any horizontal point on the graph. As shown in Fig. 14.71, when the cursor is set on 101.5 ms **(x1),** the peak value of the current curve is 159.05 mA **(y1)**. A second cursor appears in blue with a number **2** in the inverted arrowhead that can also be moved with a left click on the number **2** at the top of the line. If set at 101.75 ms **(x2),** it has a minimum value of −5.18 mA **(y2)**, the smallest value available for the calculated data points. Note that the difference between horizontal time values **dx** = 252 μs = 0.25 ms which is ¼ of the period of the wave (at 1 ms).

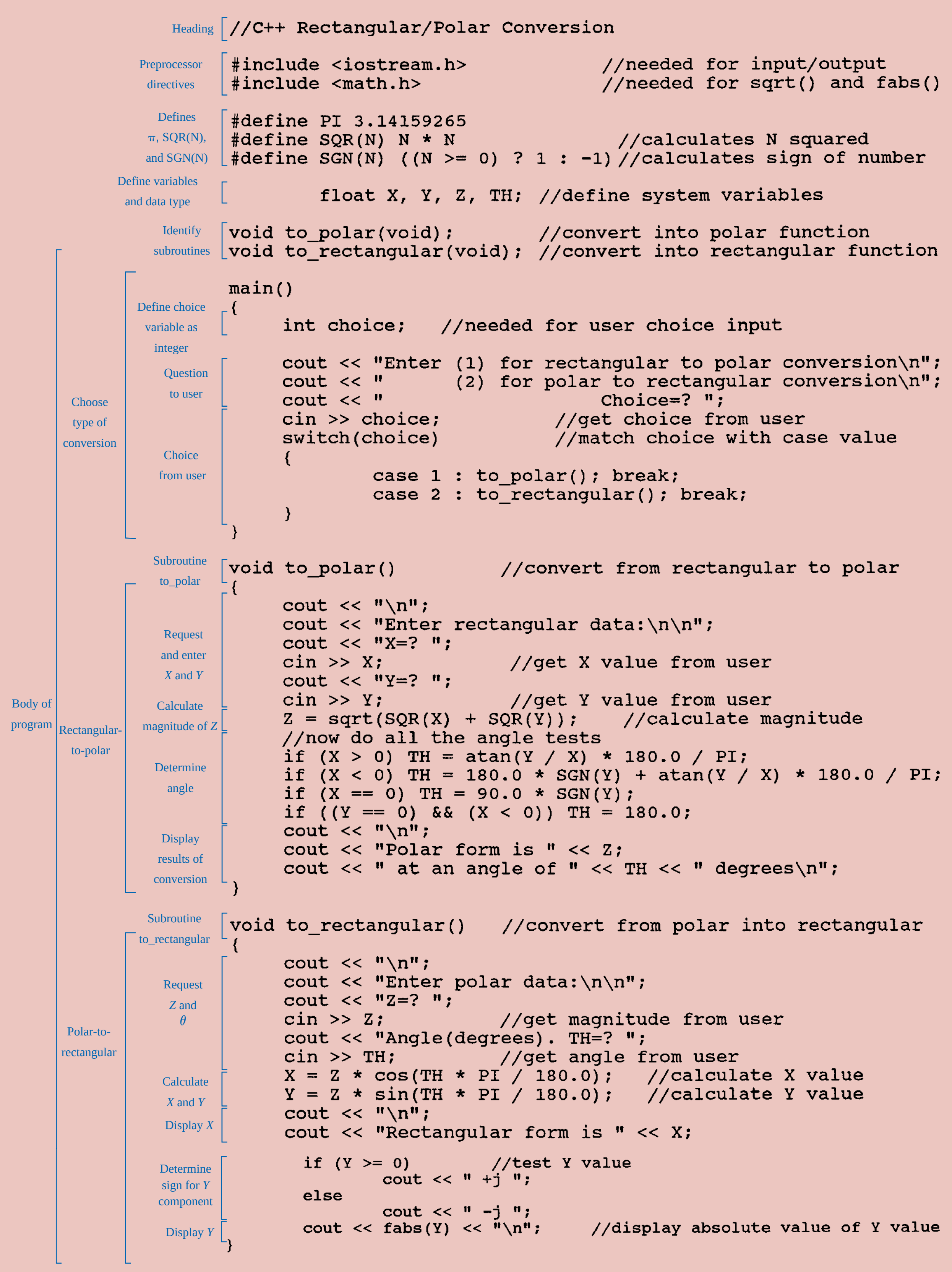
C++

The versatility of the C++ programming language is clearly demonstrated by the following program designed to perform conversions between the polar and rectangular forms. Comments are provided on the right side of the program to help identify the function of specific lines or sections of the program. Recall that any comments to the right of the parallel slash bars // are ignored by the compiler. In this case the file *math.h* must be added to the preprocessor directive list, as shown in Fig. 14.72, to provide the mathematical functions to be employed in the program. A complete list of operations can be found in the compiler reference manual. The *#define* directive defines the level of *PI* to be employed when called for in the program and specifies the operations to be performed when *SQR*(*N*) and *SGN*(*N*) appear. The *?* associated with the *SGN*(*N*) directive is a *conditional operator* that specifies +1 if *N* is greater than or equal to 0 and −1 if not.

Next the variables are introduced and defined as floating points. The next entry includes the term *void* to indicate that the variable *to_polar* will not return a specific numerical value when part of an execution but rather may identify a subroutine or string of words or characters. The *void* within the parentheses reveals that the variable does not have a list of parameters associated with it for possible use in an application.

As described in earlier programs the *main* ( ) defines the point at which execution will begin, with the body of *main* defined by the opening and closing braces { }. Within *main*, an integer variable *choice* is introduced to handle the integer number (1 or 2) which the user will choose in response to the question posed under *cout.* Through *cin* the user will respond with a 1 or 2, which will define the variable *choice.* The *switch* is a conditional response that will follow a path defined by the variable *choice.* The possible paths for the program to follow under *switch* are enclosed in the braces { }. Since a numerical value will determine the path, the options must begin with the word *case.* In this case, a 1 will follow the *to_polar* structured variable, and a 2 will follow the *to_rectangular* structured variable. The *break* simply marks the end of the selection process.

On a *to_polar* choice the program will move to the subroutine *void to_polar* and will convert the number to the polar form. The first six lines simply create line shifts and ask for the values of *X* and *Y*. The next line calculates the magnitude of the polar form (*Z*) using *SQR*(*N*), defined above, and the *sqrt* from the *math.h* header file. An *if* statement sensitive to the value of *X* and *Y* will then delineate which line will determine the phase angle of the polar form. The *SGN*(*N*), as introduced in the preprocessor listing, will determine the sign to be employed in the equation. The *a* preceding the tan function indicates arc tan or tan^{-1}, while *PI* is as defined above in the preprocessor section. Note also that the angles must



```
//C++ Rectangular/Polar Conversion

#include <iostream.h>            //needed for input/output
#include <math.h>                //needed for sqrt() and fabs()

#define PI 3.14159265
#define SQR(N) N * N                  //calculates N squared
#define SGN(N) ((N >= 0) ? 1 : -1)//calculates sign of number

        float X, Y, Z, TH;  //define system variables

void to_polar(void);        //convert into polar function
void to_rectangular(void);  //convert into rectangular function

main()
{
     int choice;   //needed for user choice input

     cout << "Enter (1) for rectangular to polar conversion\n";
     cout << "      (2) for polar to rectangular conversion\n";
     cout << "                   Choice=? ";
     cin >> choice;          //get choice from user
     switch(choice)          //match choice with case value
     {
             case 1 : to_polar(); break;
             case 2 : to_rectangular(); break;
     }
}

void to_polar()         //convert from rectangular to polar
{
     cout << "\n";
     cout << "Enter rectangular data:\n\n";
     cout << "X=? ";
     cin >> X;          //get X value from user
     cout << "Y=? ";
     cin >> Y;          //get Y value from user
     Z = sqrt(SQR(X) + SQR(Y));    //calculate magnitude
     //now do all the angle tests
     if (X > 0) TH = atan(Y / X) * 180.0 / PI;
     if (X < 0) TH = 180.0 * SGN(Y) + atan(Y / X) * 180.0 / PI;
     if (X == 0) TH = 90.0 * SGN(Y);
     if ((Y == 0) && (X < 0)) TH = 180.0;
     cout << "\n";
     cout << "Polar form is " << Z;
     cout << " at an angle of " << TH << " degrees\n";
}

void to_rectangular()   //convert from polar into rectangular
{
     cout << "\n";
     cout << "Enter polar data:\n\n";
     cout << "Z=? ";
     cin >> Z;         //get magnitude from user
     cout << "Angle(degrees). TH=? ";
     cin >> TH;        //get angle from user
     X = Z * cos(TH * PI / 180.0);   //calculate X value
     Y = Z * sin(TH * PI / 180.0);   //calculate Y value
     cout << "\n";
     cout << "Rectangular form is " << X;

      if (Y >= 0)       //test Y value
             cout << " +j ";
      else
             cout << " -j ";
      cout << fabs(Y) << "\n";     //display absolute value of Y value
}
```

FIG. 14.72

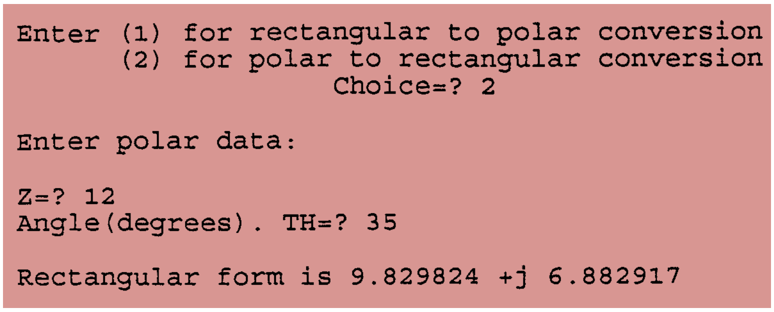
C++ program for complex number conversions.

first be converted to radians by multiplying by the ratio 180°/π. Once determined, the polar form is printed out using the *cout* statements.

Choosing the *to_rectangular* structured variable will cause the program to bypass the above subroutine and move directly to the polar-to-rectangular-conversion sequence. Again, the first six lines simply ask for the components of the polar form. The real and imaginary parts are then calculated and the results printed out. Note the *if-else* statement required to associate the properly signed *j* with the imaginary part.

In an effort to clearly identify the major components of the program, brackets have been added at the edge of the program with a short description of the function performed. As mentioned earlier, do not be concerned if a number of questions arise about the program structure or specific commands or statements. The purpose here is simply to introduce the basic format of the C++ programming language and not to provide all the details required to write your own programs.

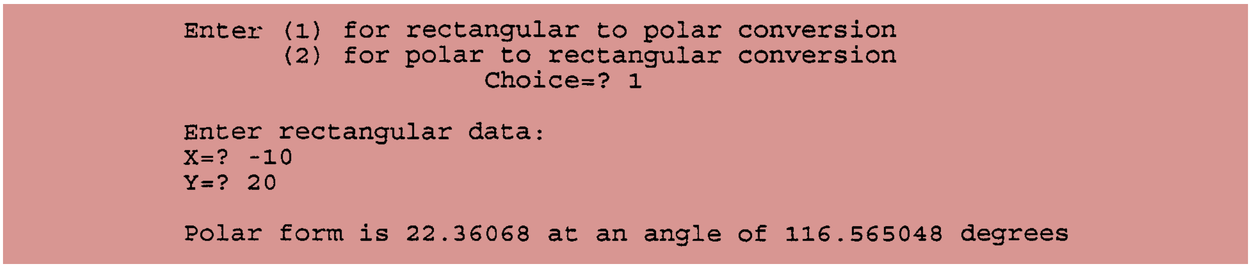
Two runs of the program have been provided in Figs. 14.73 and 14.74, one for a polar-to-rectangular conversion and the other for a rectangular-to-polar conversion. Note in each case the result of the *cout* and *cin* statements and in general the clean, clear, and direct format of the resulting output.



```
Enter (1) for rectangular to polar conversion
      (2) for polar to rectangular conversion
                   Choice=? 2

Enter polar data:

Z=? 12
Angle(degrees). TH=? 35

Rectangular form is 9.829824 +j 6.882917
```

FIG. 14.73

Polar-to-rectangular conversion using the C++ program of Fig. 14.72.



```
Enter (1) for rectangular to polar conversion
      (2) for polar to rectangular conversion
                   Choice=? 1

Enter rectangular data:
X=? -10
Y=? 20

Polar form is 22.36068 at an angle of 116.565048 degrees
```

FIG. 14.74

Rectangular-to-polar conversion using the C++ program of Fig. 14.72.

PROBLEMS

SECTION 14.2 The Derivative

1. Plot the following waveform versus time showing one clear, complete cycle. Then determine the derivative of the waveform using Eq. (14.1), and sketch one complete cycle of the derivative directly under the original waveform. Compare the magnitude of the derivative at various points versus the slope of the original sinusoidal function.

$$v = 1 \sin 3.14t$$

2. Repeat Problem 1 for the following sinusoidal function, and compare results. In particular, determine the frequency of the waveforms of Problems 1 and 2, and compare the magnitude of the derivative.

$$v = 1 \sin 15.71t$$

3. What is the derivative of each of the following sinusoidal expressions?

a. $10 \sin 377t$ **b.** $0.6 \sin(754t + 20°)$
c. $\sqrt{2}\ 20 \sin(157t - 20°)$ **d.** $-200 \sin(t + 180°)$

SECTION 14.3 Response of Basic *R*, *L*, and *C* Elements to a Sinusoidal Voltage or Current

4. The voltage across a 5-Ω resistor is as indicated. Find the sinusoidal expression for the current. In addition, sketch the *v* and *i* sinusoidal waveforms on the same axis.

a. $150 \sin 377t$ **b.** $30 \sin(377t + 20°)$
c. $40 \cos(\omega t + 10°)$ **d.** $-80 \sin(\omega t + 40°)$

5. The current through a 7-kΩ resistor is as indicated. Find the sinusoidal expression for the voltage. In addition, sketch the *v* and *i* sinusoidal waveforms on the same axis.

a. $0.03 \sin 754t$
b. $2 \times 10^{-3} \sin(400t - 120°)$
c. $6 \times 10^{-6} \cos(\omega t - 2°)$
d. $-0.004 \cos(\omega t - 90°)$

6. Determine the inductive reactance (in ohms) of a 2-H coil for

a. dc

and for the following frequencies:

b. 25 Hz **c.** 60 Hz
d. 2000 Hz **e.** 100,000 Hz

7. Determine the inductance of a coil that has a reactance of

a. 20 Ω at $f = 2$ Hz.
b. 1000 Ω at $f = 60$ Hz.
c. 5280 Ω at $f = 1000$ Hz.

8. Determine the frequency at which a 10-H inductance has the following inductive reactances:

a. 50 Ω **b.** 3770 Ω
c. 15.7 kΩ **d.** 243 Ω

9. The current through a 20-Ω inductive reactance is given. What is the sinusoidal expression for the voltage? Sketch the *v* and *i* sinusoidal waveforms on the same axis.

a. $i = 5 \sin \omega t$ **b.** $i = 0.4 \sin(\omega t + 60°)$
c. $i = -6 \sin(\omega t - 30°)$ **d.** $i = 3 \cos(\omega t + 10°)$

10. The current through a 0.1-H coil is given. What is the sinusoidal expression for the voltage?

a. $30 \sin 30t$
b. $0.006 \sin 377t$
c. $5 \times 10^{-6} \sin(400t + 20°)$
d. $-4 \cos(20t - 70°)$

11. The voltage across a 50-Ω inductive reactance is given. What is the sinusoidal expression for the current? Sketch the *v* and *i* sinusoidal waveforms on the same set of axes.

a. $50 \sin \omega t$ **b.** $30 \sin(\omega t + 20°)$
c. $40 \cos(\omega t + 10°)$ **d.** $-80 \sin(377t + 40°)$

12. The voltage across a 0.2-H coil is given. What is the sinusoidal expression for the current?

a. $1.5 \sin 60t$
b. $0.016 \sin(t + 4°)$
c. $-4.8 \sin(0.05t + 50°)$
d. $9 \times 10^{-3} \cos(377t + 360°)$

13. Determine the capacitive reactance (in ohms) of a 5-μF capacitor for

a. dc

and for the following frequencies:

b. 60 Hz **c.** 120 Hz
d. 1800 Hz **e.** 24,000 Hz

14. Determine the capacitance in microfarads if a capacitor has a reactance of

a. 250 Ω at $f = 60$ Hz.
b. 55 Ω at $f = 312$ Hz.
c. 10 Ω at $f = 25$ Hz.

15. Determine the frequency at which a 50-μF capacitor has the following capacitive reactances:

a. 342 Ω **b.** 684 Ω
c. 171 Ω **d.** 2000 Ω

16. The voltage across a 2.5-Ω capacitive reactance is given. What is the sinusoidal expression for the current? Sketch the *v* and *i* sinusoidal waveforms on the same set of axes.

a. $100 \sin \omega t$ **b.** $0.4 \sin(\omega t + 20°)$
c. $8 \cos(\omega t + 10°)$ **d.** $-70 \sin(\omega t + 40°)$

17. The voltage across a 1-μF capacitor is given. What is the sinusoidal expression for the current?

a. $30 \sin 200t$ **b.** $90 \sin 377t$
c. $-120 \sin(374t + 30°)$ **d.** $70 \cos(800t - 20°)$

18. The current through a 10-Ω capacitive reactance is given. Write the sinusoidal expression for the voltage. Sketch the *v* and *i* sinusoidal waveforms on the same set of axes.

a. $i = 50 \sin \omega t$ **b.** $i = 40 \sin(\omega t + 60°)$
c. $i = -6 \sin(\omega t - 30°)$ **d.** $i = 3 \cos(\omega t + 10°)$

19. The current through a 0.5-μF capacitor is given. What is the sinusoidal expression for the voltage?

a. $0.20 \sin 300t$ **b.** $0.007 \sin 377t$
c. $0.048 \cos 754t$ **d.** $0.08 \sin(1600t - 80°)$

***20.** For the following pairs of voltages and currents, indicate whether the element involved is a capacitor, an inductor, or a resistor, and the value of C, L, or R if sufficient data are given:

a. $v = 550 \sin(377t + 40°)$
$i = 11 \sin(377t - 50°)$

b. $v = 36 \sin(754t + 80°)$
$i = 4 \sin(754t + 170°)$

c. $v = 10.5 \sin(\omega t + 13°)$
$i = 1.5 \sin(\omega t + 13°)$

***21.** Repeat Problem 20 for the following pairs of voltages and currents:

a. $v = 2000 \sin \omega t$
$i = 5 \cos \omega t$

b. $v = 80 \sin(157t + 150°)$
$i = 2 \sin(157t + 60°)$

c. $v = 35 \sin(\omega t - 20°)$
$i = 7 \cos(\omega t - 110°)$

SECTION 14.4 Frequency Response of the Basic Elements

22. Plot X_L versus frequency for a 5-mH coil using a frequency range of zero to 100 kHz on a linear scale.

23. Plot X_C versus frequency for a 1-μF capacitor using a frequency range of zero to 10 kHz on a linear scale.

24. At what frequency will the reactance of a 1-μF capacitor equal the resistance of a 2-kΩ resistor?

25. The reactance of a coil equals the resistance of a 10-kΩ resistor at a frequency of 5 kHz. Determine the inductance of the coil.

26. Determine the frequency at which a 1-μF capacitor and a 10-mH inductor will have the same reactance.

27. Determine the capacitance required to establish a capacitive reactance that will match that of a 2-mH coil at a frequency of 50 kHz.

SECTION 14.5 Average Power and Power Factor

28. Find the average power loss in watts for each set in Problem 20.

29. Find the average power loss in watts for each set in Problem 21.

***30.** Find the average power loss and power factor for each of the circuits whose input current and voltage are as follows:

a. $v = 60 \sin(\omega t + 30°)$
$i = 15 \sin(\omega t + 60°)$

b. $v = -50 \sin(\omega t - 20°)$
$i = -2 \sin(\omega t + 40°)$

c. $v = 50 \sin(\omega t + 80°)$
$i = 3 \cos(\omega t + 20°)$

d. $v = 75 \sin(\omega t - 5°)$
$i = 0.08 \sin(\omega t - 35°)$

31. If the current through and voltage across an element are $i = 8 \sin(\omega t + 40°)$ and $v = 48 \sin(\omega t + 40°)$, respectively, compute the power by I^2R, $(V_m I_m/2) \cos \theta$, and $VI \cos \theta$, and compare answers.

32. A circuit dissipates 100 W (average power) at 150 V (effective input voltage) and 2 A (effective input current). What is the power factor? Repeat if the power is 0 W; 300 W.

***33.** The power factor of a circuit is 0.5 lagging. The power delivered in watts is 500. If the input voltage is $50 \sin(\omega t + 10°)$, find the sinusoidal expression for the input current.

34. In Fig. 14.75, $e = 30 \sin(377t + 20°)$.

a. What is the sinusoidal expression for the current?

b. Find the power loss in the circuit.

c. How long (in seconds) does it take the current to complete six cycles?

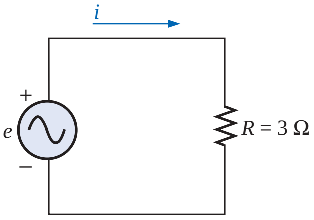


FIG. 14.75
Problem 34.

35. In Fig. 14.76, $e = 100 \sin(157t + 30°)$.

a. Find the sinusoidal expression for i.

b. Find the value of the inductance L.

c. Find the average power loss by the inductor.

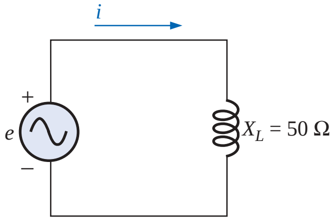


FIG. 14.76
Problem 35.

36. In Fig. 14.77, $i = 3 \sin(377t - 20°)$.

a. Find the sinusoidal expression for e.

b. Find the value of the capacitance C in microfarads.

c. Find the average power loss in the capacitor.

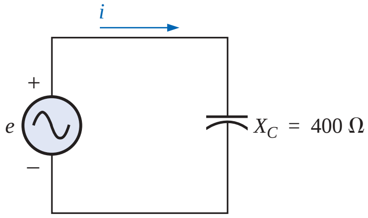


FIG. 14.77
Problem 36.

***37.** For the network of Fig. 14.78 and the applied signal:

- **a.** Determine i_1 and i_2.
- **b.** Find i_s.

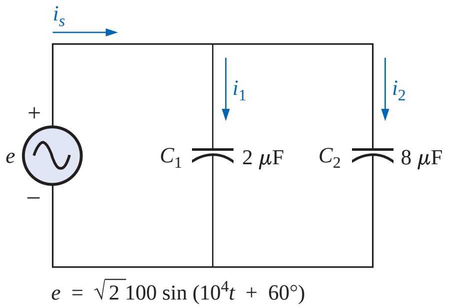


FIG. 14.78
Problem 37.

***38.** For the network of Fig. 14.79 and the applied source:

- **a.** Determine the source voltage v_s.
- **b.** Find the currents i_1 and i_2.

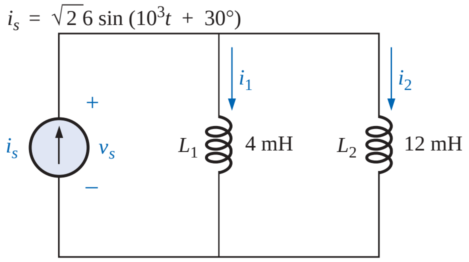


FIG. 14.79
Problem 38.

SECTION 14.9 Conversion between Forms

39. Convert the following from rectangular to polar form:

- **a.** $4 + j3$
- **b.** $2 + j2$
- **c.** $3.5 + j16$
- **d.** $100 + j800$
- **e.** $1000 + j400$
- **f.** $0.001 + j0.0065$
- **g.** $7.6 - j9$
- **h.** $-8 + j4$
- **i.** $-15 - j60$
- **j.** $+78 - j65$
- **k.** $-2400 + j3600$
- **l.** $5 \times 10^{-3} - j25 \times 10^{-3}$

40. Convert the following from polar to rectangular form:

- **a.** $6 \angle 30°$
- **b.** $40 \angle 80°$
- **c.** $7400 \angle 70°$
- **d.** $4 \times 10^{-4} \angle 8°$
- **e.** $0.04 \angle 80°$
- **f.** $0.0093 \angle 23°$
- **g.** $65 \angle 150°$
- **h.** $1.2 \angle 135°$
- **i.** $500 \angle 200°$
- **j.** $6320 \angle -35°$
- **k.** $7.52 \angle -125°$
- **l.** $0.008 \angle 310°$

41. Convert the following from rectangular to polar form:

- **a.** $1 + j15$
- **b.** $60 + j5$
- **c.** $0.01 + j0.3$
- **d.** $100 - j2000$
- **e.** $-5.6 + j86$
- **f.** $-2.7 - j38.6$

42. Convert the following from polar to rectangular form:

- **a.** $13 \angle 5°$
- **b.** $160 \angle 87°$
- **c.** $7 \times 10^{-6} \angle 2°$
- **d.** $8.7 \angle 177°$
- **e.** $76 \angle -4°$
- **f.** $396 \angle +265°$

SECTION 14.10 Mathematical Operations with Complex Numbers

Perform the following operations.

43. Addition and subtraction (express your answers in rectangular form):

- **a.** $(4.2 + j6.8) + (7.6 + j0.2)$
- **b.** $(142 + j7) + (9.8 + j42) + (0.1 + j0.9)$
- **c.** $(4 \times 10^{-6} + j76) + (7.2 \times 10^{-7} - j5)$
- **d.** $(9.8 + j6.2) - (4.6 + j4.6)$
- **e.** $(167 + j243) - (-42.3 - j68)$
- **f.** $(-36.0 + j78) - (-4 - j6) + (10.8 - j72)$
- **g.** $6 \angle 20° + 8 \angle 80°$
- **h.** $42 \angle 45° + 62 \angle 60° - 70 \angle 120°$

44. Multiplication [express your answers in rectangular form for parts (a) through (d), and in polar form for parts (e) through (h)]:

- **a.** $(2 + j3)(6 + j8)$
- **b.** $(7.8 + j1)(4 + j2)(7 + j6)$
- **c.** $(0.002 + j0.006)(-2 + j2)$
- **d.** $(400 - j200)(-0.01 - j0.5)(-1 + j3)$
- **e.** $(2 \angle 60°)(4 \angle 22°)$
- **f.** $(6.9 \angle 8°)(7.2 \angle -72°)$
- **g.** $0.002 \angle 120°)(0.5 \angle 200°)(40 \angle -60°)$
- **h.** $(540 \angle -20°)(-5 \angle 180°)(6.2 \angle 0°)$

45. Division (express your answers in polar form):

- **a.** $(42 \angle 10°)/(7 \angle 60°)$
- **b.** $(0.006 \angle 120°)/(30 \angle -20°)$
- **c.** $(4360 \angle -20°)/(40 \angle 210°)$
- **d.** $(650 \angle -80°)/(8.5 \angle 360°)$
- **e.** $(8 + j8)/(2 + j2)$
- **f.** $(8 + j42)/(-6 + j60)$
- **g.** $(0.05 + j0.25)/(8 - j60)$
- **h.** $(-4.5 - j6)/(0.1 - j0.4)$

***46.** Perform the following operations (express your answers in rectangular form):

- **a.** $\dfrac{(4 + j3) + (6 - j8)}{(3 + j3) - (2 + j3)}$
- **b.** $\dfrac{8 \angle 60°}{(2 \angle 0°) + (100 + j100)}$
- **c.** $\dfrac{(6 \angle 20°)(120 \angle -40°)(3 + j4)}{2 \angle -30°}$
- **d.** $\dfrac{(0.4 \angle 60°)^2(300 \angle 40°)}{3 + j9}$
- **e.** $\left(\dfrac{1}{(0.02 \angle 10°)^2}\right)\left(\dfrac{2}{j}\right)^3\left(\dfrac{1}{6^2 - j\sqrt{900}}\right)$

***47.** **a.** Determine a solution for x and y if

$$(x + j4) + (3x + jy) - j7 = 16 \angle 0°$$

b. Determine x if

$$(10 \angle 20°)(x \angle -60°) = 30.64 - j25.72$$

c. Determine a solution for x and y if

$$(5x + j10)(2 - jy) = 90 - j70$$

d. Determine θ if

$$\frac{80 \angle 0°}{20 \angle \theta} = 3.464 - j2$$

SECTION 14.12 Phasors

48. Express the following in phasor form:

a. $\sqrt{2}(100) \sin(\omega t + 30°)$
b. $\sqrt{2}(0.25) \sin(157t - 40°)$
c. $100 \sin(\omega t - 90°)$
d. $42 \sin(377t + 0°)$
e. $6 \times 10^{-6} \cos \omega t$
f. $3.6 \times 10^{-6} \cos(754t - 20°)$

49. Express the following phasor currents and voltages as sine waves if the frequency is 60 Hz:

a. $\mathbf{I} = 40 \text{ A} \angle 20°$ **b.** $\mathbf{V} = 120 \text{ V} \angle 0°$
c. $\mathbf{I} = 8 \times 10^{-3} \text{ A} \angle 120°$ **d.** $\mathbf{V} = 5 \text{ V} \angle 90°$
e. $\mathbf{I} = 1200 \text{ A} \angle -120°$ **f.** $\mathbf{V} = \dfrac{6000}{\sqrt{2}} \text{ V} \angle -180°$

50. For the system of Fig. 14.80, find the sinusoidal expression for the unknown voltage v_a if

$$e_{\text{in}} = 60 \sin(377t + 20°)$$
$$v_b = 20 \sin 377t$$

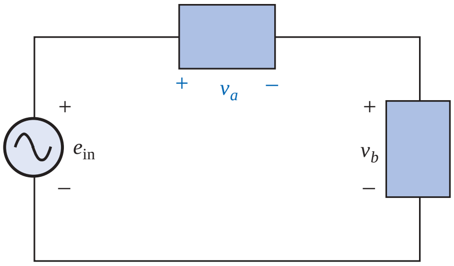


FIG. 14.80
Problem 50.

51. For the system of Fig. 14.81, find the sinusoidal expression for the unknown current i_1 if

$$i_s = 20 \times 10^{-6} \sin(\omega t + 90°)$$
$$i_2 = 6 \times 10^{-6} \sin(\omega t - 60°)$$

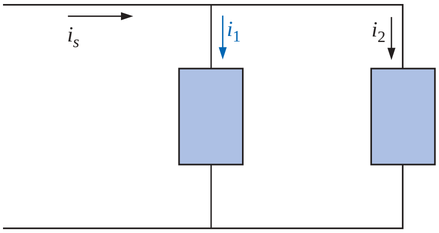


FIG. 14.81
Problem 51.

52. Find the sinusoidal expression for the applied voltage e for the system of Fig. 14.82 if

$$v_a = 60 \sin(\omega t + 30°)$$
$$v_b = 30 \sin(\omega t - 30°)$$
$$v_c = 40 \sin(\omega t + 120°)$$

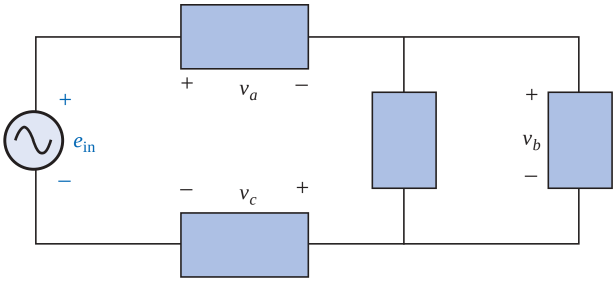


FIG. 14.82
Problem 52.

53. Find the sinusoidal expression for the current i_s for the system of Fig. 14.83 if

$$i_1 = 6 \times 10^{-3} \sin(377t + 180°)$$
$$i_2 = 8 \times 10^{-3} \sin 377t$$
$$i_3 = 2i_2$$

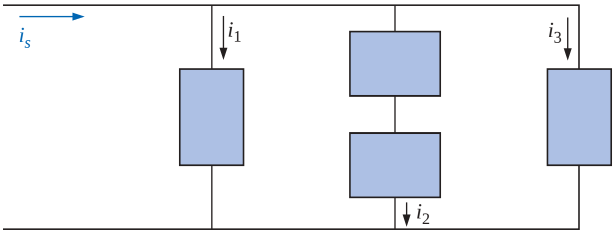


FIG. 14.83
Problem 53.

SECTION 14.13 Computer Analysis

PSpice or Electronics Workbench

54. Plot i_C and v_C versus time for the network of Fig. 14.69 for two cycles if the frequency is 0.2 kHz.

55. Plot the magnitude and phase angle of the current i_C versus frequency (100 Hz to 100 kHz) for the network of Fig. 14.69.

56. Plot the total impedance of the configuration of Fig. 14.26 versus frequency (100 kHz to 100 MHz) for the following parameter values: $C = 0.1\ \mu\text{F}$, $L_s = 0.2\ \mu\text{H}$, $R_s = 2\ \text{M}\Omega$, and $R_p = 100\ \text{M}\Omega$. For what frequency range is the capacitor "capacitive"?

Programming Language (C++, QBASIC, Pascal, etc.)

57. Given a sinusoidal function, write a program to print out the derivative.

58. Given the sinusoidal expression for the current, determine the expression for the voltage across a resistor, a capacitor, or an inductor, depending on the element involved. In other words, the program will ask which element is to be investigated and will then request the pertinent data to obtain the mathematical expression for the sinusoidal voltage.

59. Write a program to tabulate the reactance versus frequency for an inductor or a capacitor for a specified frequency range.

60. Given the sinusoidal expression for the voltage and current of a load, write a program to determine the average power and power factor.

61. Given two sinusoidal functions, write a program to convert each to the phasor domain, add the two, and print out the sum in the phasor and time domains.

GLOSSARY

Average or **real power** The power delivered to and dissipated by the load over a full cycle.

Complex conjugate A complex number defined by simply changing the sign of an imaginary component of a complex number in the rectangular form.

Complex number A number that represents a point in a two-dimensional plane located with reference to two distinct axes. It defines a vector drawn from the origin to that point.

Derivative The instantaneous rate of change of a function with respect to time or another variable.

Leading and **lagging power factors** An indication of whether a network is primarily capacitive or inductive in nature. Leading power factors are associated with capacitive networks, and lagging power factors with inductive networks.

Phasor A radius vector that has a constant magnitude at a fixed angle from the positive real axis and that represents a sinusoidal voltage or current in the vector domain.

Phasor diagram A "snapshot" of the phasors that represent a number of sinusoidal waveforms at $t = 0$.

Polar form A method of defining a point in a complex plane that includes a single magnitude to represent the distance from the origin, and an angle to reflect the counterclockwise distance from the positive real axis.

Power factor (F_p) An indication of how reactive or resistive an electrical system is. The higher the power factor, the greater the resistive component.

Reactance The opposition of an inductor or a capacitor to the flow of charge that results in the continual exchange of energy between the circuit and magnetic field of an inductor or the electric field of a capacitor.

Reciprocal A format defined by 1 divided by the complex number.

Rectangular form A method of defining a point in a complex plane that includes the magnitude of the real component and the magnitude of the imaginary component, the latter component being defined by an associated letter *j*.