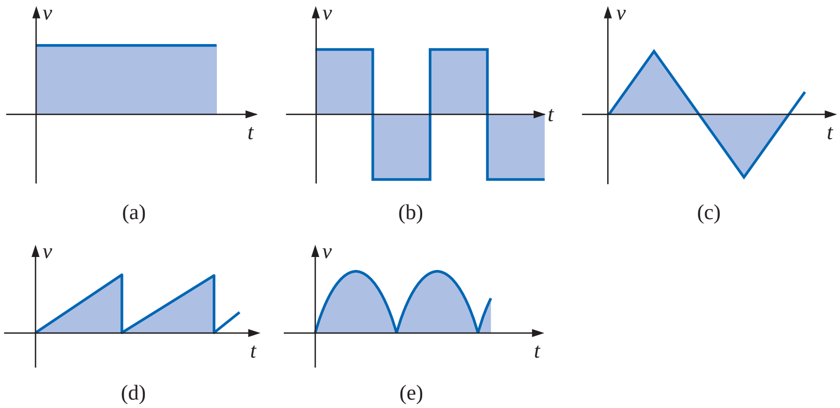
25

Nonsinusoidal Circuits

25.1 INTRODUCTION

Any waveform that differs from the basic description of the sinusoidal waveform is referred to as **nonsinusoidal.** The most obvious and familiar are the dc, square-wave, triangular, sawtooth, and rectified waveforms of Fig. 25.1.



(a) (b) (c)

(d) (e)

FIG. 25.1
Common nonsinusoidal waveforms: (a) dc; (b) square-wave; (c) triangular; (d) sawtooth; (e) rectified.

The output of many electrical and electronic devices will be nonsinusoidal, even though the applied signal may be purely sinusoidal. For example, the network of Fig. 25.2 employs a diode to clip off the negative portion of the applied signal in a process called *half-wave rectification,* which is used in the development of dc levels from a sinusoidal input. You will find in your electronics courses that the diode is similar to a mechanical switch, but it is different because it can conduct current in only one direction. The output waveform is definitely nonsinusoidal, but note that it has the same period as the applied signal and matches the input for half the period.

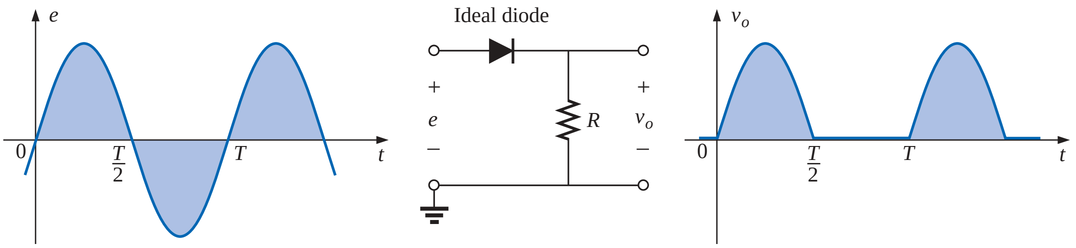


FIG. 25.2
Half-wave rectifier producing a nonsinusoidal waveform.

This chapter will demonstrate how a nonsinusoidal waveform like the output of Fig. 25.2 can be represented by a series of terms. You will also learn how to determine the response of a network to such an input.

25.2 FOURIER SERIES

Fourier series refers to a series of terms, developed in 1826 by Baron Jean Fourier (Fig. 25.3), that can be used to represent a nonsinusoidal periodic waveform. In the analysis of these waveforms, we solve for each term in the Fourier series:

$$\boxed{\begin{aligned} f(t) = \underbrace{A_0}_{\substack{\text{dc or}\\ \text{average value}}} &+ \underbrace{A_1 \sin \omega t + A_2 \sin 2\omega t + A_3 \sin 3\omega t + \cdots + A_n \sin n\omega t}_{\text{sine terms}} \\ &+ \underbrace{B_1 \cos \omega t + B_2 \cos 2\omega t + B_3 \cos 3\omega t + \cdots + B_n \cos n\omega t}_{\text{cosine terms}} \end{aligned}} \tag{25.1}$$

Depending on the waveform, a large number of these terms may be required to approximate the waveform closely for the purpose of circuit analysis.

As shown in Eq. (25.1), the Fourier series has three basic parts. The first is the dc term A_0, which is the average value of the waveform over one full cycle. The second is a series of sine terms. There are no restrictions on the values or relative values of the amplitudes of these sine terms, but each will have a frequency that is an integer multiple of the frequency of the first sine term of the series. The third part is a series of cosine terms. There are again *no* restrictions on the values or relative values of the amplitudes of these cosine terms, but each will have a frequency that is an integer multiple of the frequency of the first cosine term of the series. For a particular waveform, it is quite possible that all of the sine *or* cosine terms are zero. Characteristics of this type can be determined by simply examining the nonsinusoidal waveform and its position on the horizontal axis.

The first term of the sine and cosine series is called the **fundamental component.** It represents the minimum frequency term required to represent a particular waveform, and it also has the same frequency as the waveform being represented. A fundamental term, therefore, must be present in any Fourier series representation. The other terms with higher-order frequencies (integer multiples of the fundamental) are called the **harmonic terms.** A term that has a frequency equal to twice the fundamental is the second harmonic; three times, the third harmonic; and so on.

French (Auxerre, Grenoble, Paris)
(1768–1830)
Mathematician, Egyptologist, and **Administrator**
Professor of Mathematics, École Polytechnique

Courtesy of the Smithsonian Institution Photo No. 56,822

Best known for an infinite mathematical series of sine and cosine terms called the *Fourier series* which he used to show how the conduction of heat in solids can be analyzed and defined. Although he was primarily a mathematician, a great deal of Fourier's work revolved around real-world physical occurrences such as heat transfer, sunspots, and the weather. He joined the École Polytechnique in Paris as a faculty member when the institute first opened. Napoleon requested his aid in the research of Egyptian antiquities, resulting in a three-year stay in Egypt as Secretary of the Institut d'Égypte. Napoleon made him a baron in 1809, and he was elected to the Académie des Sciences in 1817.

FIG. 25.3
Baron Jean Fourier.

Average Value: A_0

The dc term of the Fourier series is the average value of the waveform over one full cycle. If the net area above the horizontal axis equals that below in one full period, $A_0 = 0$, and the dc term does not appear in the expansion. If the area above the axis is greater than that below over one full cycle, A_0 is positive and will appear in the Fourier series representation. If the area below the axis is greater, A_0 is negative and will appear with the negative sign in the expansion.

Odd Function (Point Symmetry)

***If a waveform is such that its value for $+t$ is the negative of that for $-t$, it is called an odd function or is said to have* point symmetry.**

Figure 25.4(a) is an example of a waveform with point symmetry. Note that the waveform has a peak value at t_1 that matches the magnitude (with the opposite sign) of the peak value at $-t_1$. For waveforms of this type, all the parameters $B_{1\rightarrow\infty}$ of Eq. (25.1) will be zero. In fact,

waveforms with point symmetry can be fully described by just the dc and sine terms of the Fourier series.

Note in Fig. 25.4(b) that a sine wave is an odd function with point symmetry.

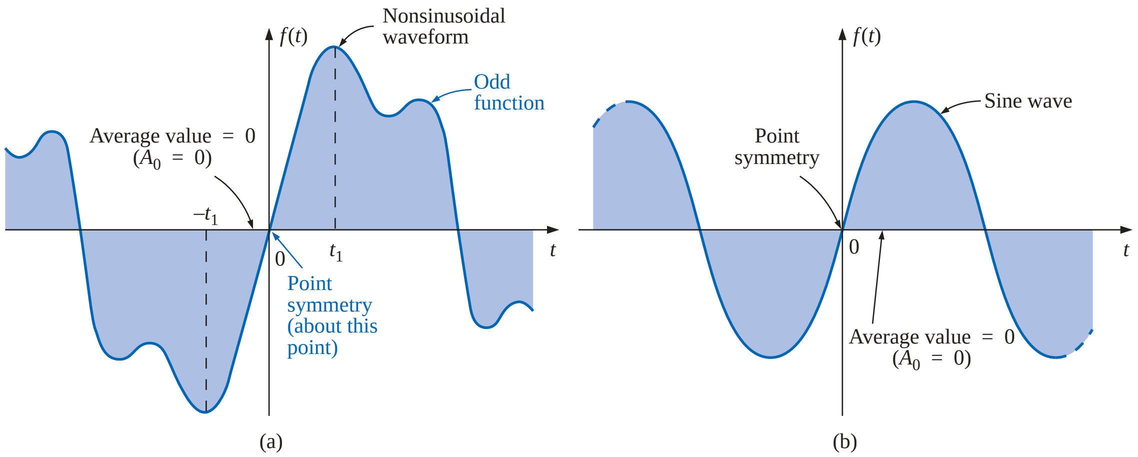


FIG. 25.4
Point symmetry.

For both waveforms of Fig. 25.4, the following mathematical relationship is true:

$$\boxed{f(t) = -f(-t)} \qquad \text{(odd function)} \qquad \textbf{(25.2)}$$

In words, it states that the magnitude of the function at $+t$ is equal to the negative of the magnitude at $-t$ [t_1 in Fig. 25.4(a)].

Even Function (Axis Symmetry)

***If a waveform is symmetric about the vertical axis, it is called an even function or is said to have* axis symmetry.**

Figure 25.5(a) is an example of such a waveform. Note that the value of the function at t_1 is equal to the value at $-t_1$. For waveforms of this type, all the parameters $A_{1\rightarrow\infty}$ will be zero. In fact,

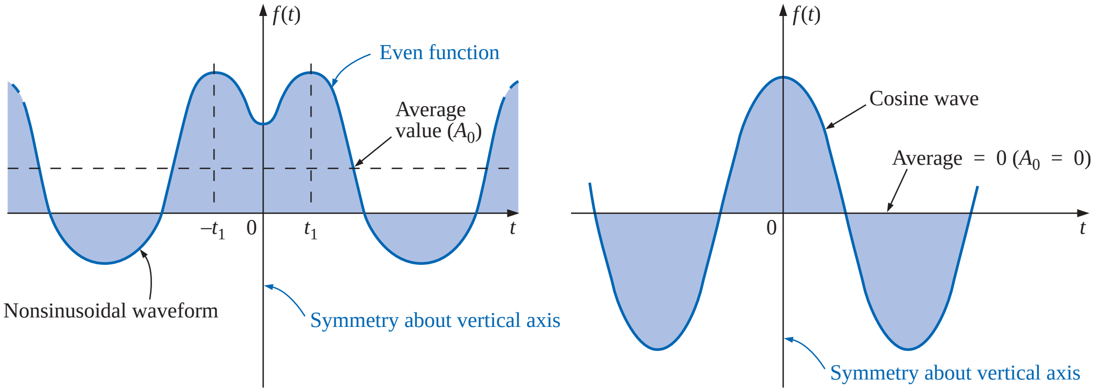
waveforms with axis symmetry can be fully described by just the dc and cosine terms of the Fourier series.

Note in Fig. 25.5(b) that a cosine wave is an even function with axis symmetry.

For both waveforms of Fig. 25.5, the following mathematical relationship is true:

$$\boxed{f(t) = f(-t)} \qquad \text{(even function)} \qquad \mathbf{(25.3)}$$

In words, it states that the magnitude of the function is the same at $+t_1$ as at $-t$ [t_1 in Fig. 25.5(a)].



(a) (b)

FIG. 25.5
Axis symmetry.

Mirror or Half-Wave Symmetry

If a waveform has* half-wave *or* mirror symmetry *as demonstrated by the waveform of Fig. 25.6, the* even harmonics *of the series of sine and cosine terms will be zero.

In functional form the waveform must satisfy the following relationship:

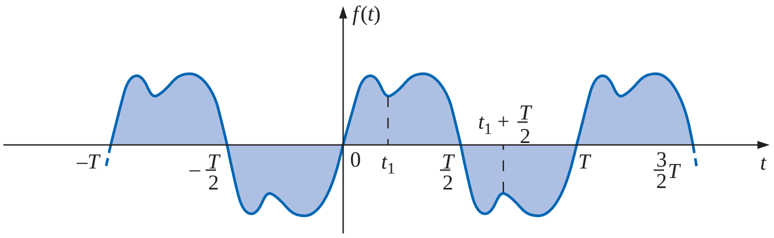


FIG. 25.6
Mirror symmetry.

$$\boxed{f(t) = -f\left(t + \frac{T}{2}\right)} \qquad \textbf{(25.4)}$$

Equation (25.4) states that the waveform encompassed in one time interval $T/2$ will repeat itself in the next $T/2$ time interval, but in the negative sense (t_1 in Fig. 25.6). For example, the waveform of Fig. 25.6 from zero to $T/2$ will repeat itself in the time interval $T/2$ to T, but below the horizontal axis.

Repetitive on the Half-Cycle

The repetitive nature of a waveform can determine whether specific harmonics will be present in the Fourier series expansion. In particular,

if a waveform is repetitive on the half-cycle as demonstrated by the waveform of Fig. 25.7, the* odd harmonics *of the series of sine and cosine terms are zero.

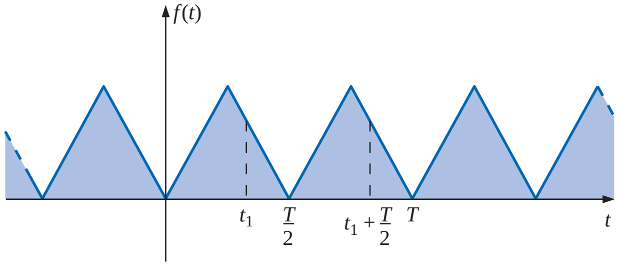


FIG. 25.7
A waveform repetitive on the half-cycle.

In functional form the waveform must satisfy the following relationship:

$$\boxed{f(t) = f\left(t + \frac{T}{2}\right)} \qquad \textbf{(25.5)}$$

Equation (25.5) states that the function repeats itself after each $T/2$ time interval (t_1 in Fig. 25.7). The waveform, however, will also repeat itself after each period T. In general, therefore, for a function of this type, if the period T of the waveform is chosen to be twice that of the minimum period ($T/2$), the odd harmonics will all be zero.

Mathematical Approach

The constants A_0, $A_{1\rightarrow n}$, $B_{1\rightarrow n}$ can be determined by using the following integral formulas:

$$\boxed{A_0 = \frac{1}{T}\int_0^T f(t)\,dt} \qquad \textbf{(25.6)}$$

$$\boxed{A_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt} \qquad \textbf{(25.7)}$$

$$\boxed{B_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt} \qquad \textbf{(25.8)}$$

These equations have been presented for recognition purposes only; they will not be used in the following analysis.

Instrumentation

FIG. 25.8
Spectrum analyzer. (Courtesy of Hewlett Packard)

There are three types of instrumentation available that will reveal the dc, fundamental, and harmonic content of a waveform: *the spectrum analyzer, wave analyzer,* and *Fourier analyzer.* The purpose of such instrumentation is not solely to determine the composition of a particular waveform but also to reveal the level of distortion that may have been introduced by a system. For instance, an amplifier may be increasing the applied signal by a factor of 50, but in the process it may have distorted the waveform in a way that is quite unnoticeable from the oscilloscope display. The amount of distortion would appear in the form of harmonics at frequencies that are multiples of the applied frequency. Each of the above instruments would reveal which frequencies are having the most impact on the distortion, permitting their removal with properly designed filters.

The *spectrum analyzer* has the appearance of an oscilloscope, as shown in Fig. 25.8, but rather than display a waveform that is voltage (vertical axis) versus time (horizontal axis), it generates a display scaled off in dB (vertical axis) versus frequency (horizontal axis). Such a display is said to be in the *frequency domain* versus the *time domain* of the standard oscilloscope. The height of the vertical line in the display of Fig. 25.8 reveals the impact of that frequency on the shape of the waveform. Spectrum analyzers are unable to provide the phase angle associated with each component.

FIG. 25.9
Wave analyzer. (Courtesy of Hewlett Packard)

The *wave analyzer* of Fig. 25.9 is a true rms voltmeter whose frequency of measurement can be changed manually. In other words, the operator works through the frequencies of interest, and the analog display will indicate the rms value of each harmonic component present. Of course, once the fundamental component is determined, the operator can quickly move through the possible harmonic levels. The wave analyzer, like the spectrum analyzer, is unable to provide the angle associated with the various components.

FIG. 25.10
Fourier analyzer. (Courtesy of Hewlett Packard)

The *Fourier analyzer* of Fig. 25.10 is similar in many respects to the spectrum analyzer except for its ability to investigate all the frequencies of interest at one time. The spectrum analyzer must review the signal one frequency at a time. The Fourier analyzer has the distinct advantage of being able to determine the phase angle of each component.

The following examples will demonstrate the use of the equations and concepts introduced thus far in this chapter.

EXAMPLE 25.1 Determine which components of the Fourier series are present in the waveforms of Fig. 25.11.

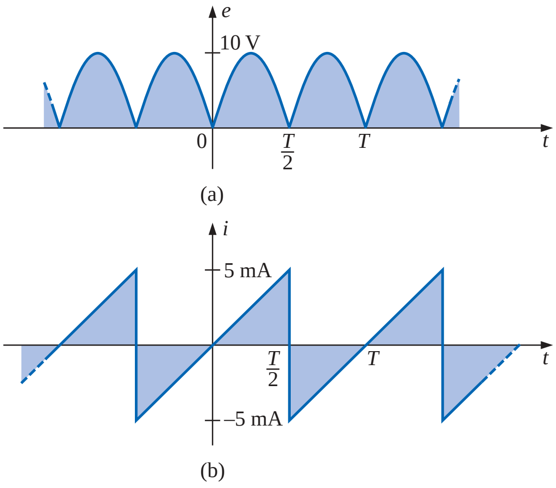


FIG. 25.11
Example 25.1.

Solutions:

a. The waveform has a net area above the horizontal axis and therefore will have **a positive dc term A_0**.

The waveform has axis symmetry, resulting in **only cosine terms** in the expansion.

The waveform has half-cycle symmetry, resulting in **only even terms** in the cosine series.

b. The waveform has the same area above and below the horizontal axis within each period, resulting in $\boldsymbol{A_0 = 0}$.

The waveform has point symmetry, resulting in **only sine terms** in the expansion.

EXAMPLE 25.2 Write the Fourier series expansion for the waveforms of Fig. 25.12.

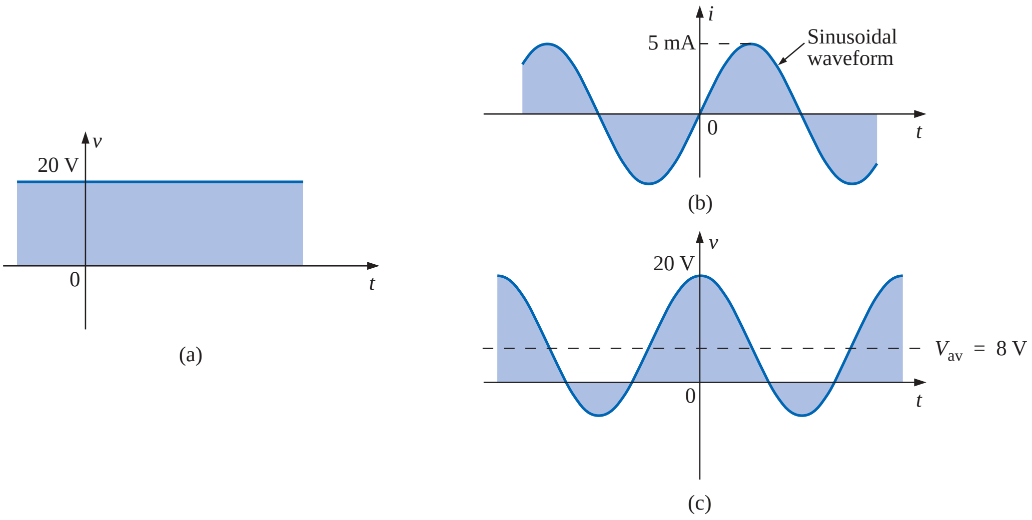


FIG. 25.12
Example 25.2.

Solutions:

a. $A_0 = 20 \qquad A_{1\rightarrow n} = 0 \qquad B_{1\rightarrow n} = 0$

$\quad v = \mathbf{20}$

b. $A_0 = 0 \qquad A_1 = 5 \times 10^{-3} \qquad A_{2\rightarrow n} = 0 \qquad B_{1\rightarrow n} = 0$

$\quad i = \mathbf{5 \times 10^{-3} \sin \omega t}$

c. $A_0 = 8 \qquad A_{1-n} = 0 \qquad B_1 = 12 \qquad B_{2\rightarrow n} = 0$

$\quad v = \mathbf{8 + 12 \cos \omega t}$

EXAMPLE 25.3 Sketch the following Fourier series expansion:

$$v = 2 + 1 \cos \alpha + 2 \sin \alpha$$

Solution: Note Fig. 25.13.

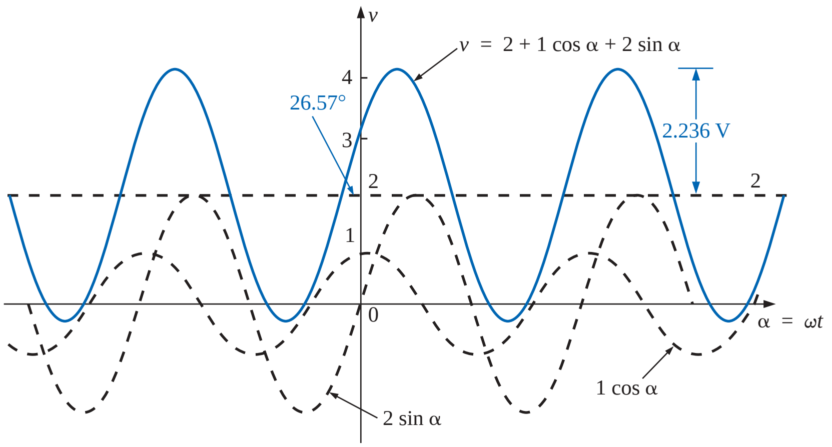


FIG. 25.13

Example 25.3.

The solution could be obtained graphically by first plotting all of the functions and then considering a sufficient number of points on the horizontal axis; or phasor algebra could be employed as follows:

$$\begin{aligned} 1 \cos \alpha + 2 \sin \alpha &= 1\text{ V} \angle 90^\circ + 2\text{ V} \angle 0^\circ = j\,1\text{ V} + 2\text{ V} \\ &= 2\text{ V} + j\,1\text{ V} = 2.236\text{ V} \angle 26.57^\circ \\ &= 2.236 \sin(\alpha + 26.57^\circ) \end{aligned}$$

and
$$v = 2 + 2.236 \sin(\alpha + 26.57^\circ)$$

which is simply the sine wave portion riding on a dc level of 2 V. That is, its positive maximum is 2 V + 2.236 V = 4.236 V, and its minimum is 2 V − 2.236 V = −0.236 V.

EXAMPLE 25.4 Sketch the following Fourier series expansion:

$$i = 1 \sin \omega t + 1 \sin 2\omega t$$

Solution: See Fig. 25.14. Note that in this case the sum of the two sinusoidal waveforms of different frequencies is *not* a sine wave. Recall that complex algebra can be applied only to waveforms having the *same* frequency. In this case the solution is obtained graphically point by point, as shown for $t = t_1$.

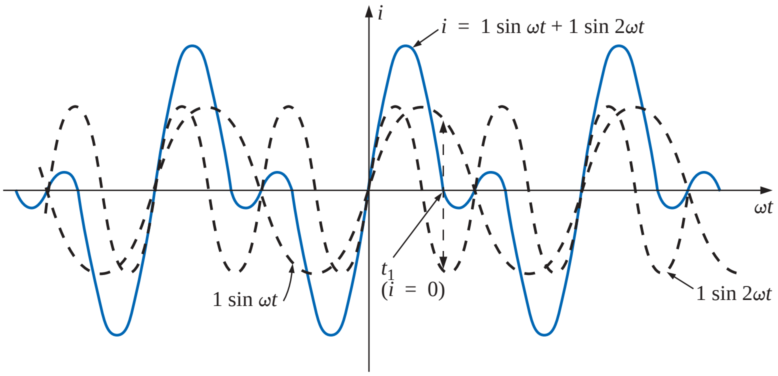


FIG. 25.14
Example 25.4.

As an additional example in the use of the Fourier series approach, consider the square wave shown in Fig. 25.15. The average value is zero, so $A_0 = 0$. It is an odd function, so all the constants $B_{1 \to n}$ equal zero; only sine terms will be present in the series expansion. Since the waveform satisfies the criteria for $f(t) = -f(t + T/2)$, the even harmonics will also be zero.

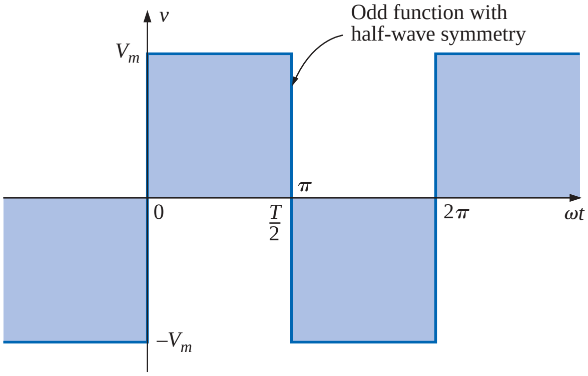


FIG. 25.15
Square wave.

The expression obtained after evaluating the various coefficients using Eq. (25.8) is

$$\boxed{v = \frac{4}{\pi} V_m \left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \frac{1}{7} \sin 7\omega t + \cdots + \frac{1}{n} \sin n\omega t \right)} \quad \textbf{(25.9)}$$

Note that the fundamental does indeed have the same frequency as that of the square wave. If we add the fundamental and third harmonics, we obtain the results shown in Fig. 25.16.

Even with only the first two terms, a few characteristics of the square wave are beginning to appear. If we add the next two terms (Fig. 25.17), the width of the pulse increases, and the number of peaks increases.

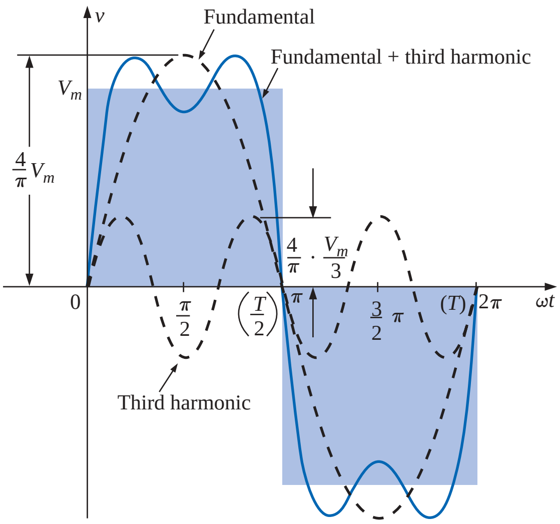


FIG. 25.16
Fundamental plus third harmonic.

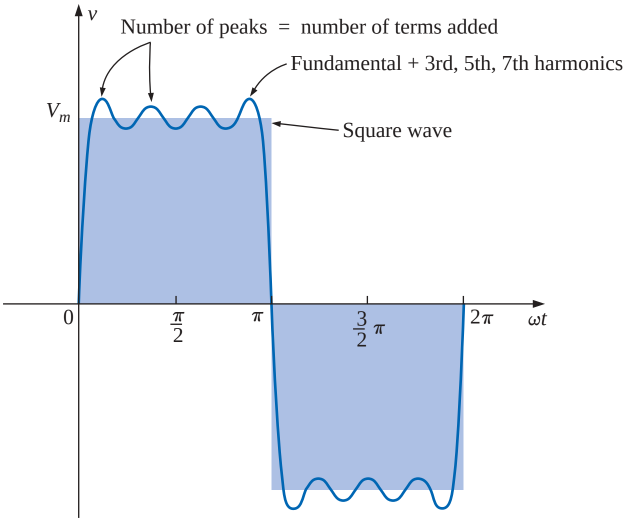


FIG. 25.17
Fundamental plus third, fifth, and seventh harmonics.

As we continue to add terms, the series will better approximate the square wave. Note, however, that the amplitude of each succeeding term diminishes to the point at which it will be negligible compared with those of the first few terms. A good approximation would be to assume that the waveform is composed of the harmonics up to and including the ninth. Any higher harmonics would be less than one-tenth the fundamental. If the waveform just described were shifted above or below the horizontal axis, the Fourier series would be altered only by a change in the dc term. Figure 25.18(c), for example, is the sum of Fig. 25.18(a) and (b). The Fourier series for the complete waveform is, therefore,

$$v = v_1 + v_2 = V_m + \text{Eq. (25.9)}$$

$$= V_m + \frac{4}{\pi}V_m\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + \cdots\right)$$

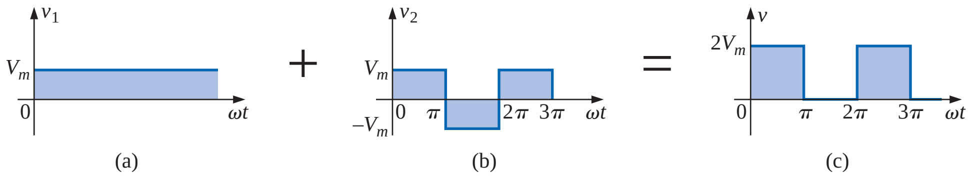


FIG. 25.18

Shifting a waveform vertically with the addition of a dc term.

and $$v = V_m\left[1 + \frac{4}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + \cdots\right)\right]$$

The equation for the half-wave rectified pulsating waveform of Fig. 25.19(b) is

$$\boxed{v_2 = 0.318V_m + 0.500V_m \sin \alpha - 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha - \cdots} \tag{25.10}$$

The waveform in Fig. 25.19(c) is the sum of the two in Fig. 25.19(a) and (b). The Fourier series for the waveform of Fig. 25.19(c) is, therefore,

$$\begin{aligned} v_T = v_1 + v_2 &= -\frac{V_m}{2} + \text{Eq. (25.10)} \\ &= -0.500V_m + 0.318V_m + 0.500V_m \sin \alpha - 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha + \cdots \end{aligned}$$

and $$v_T = -0.182V_m + 0.5V_m \sin \alpha - 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha + \cdots$$

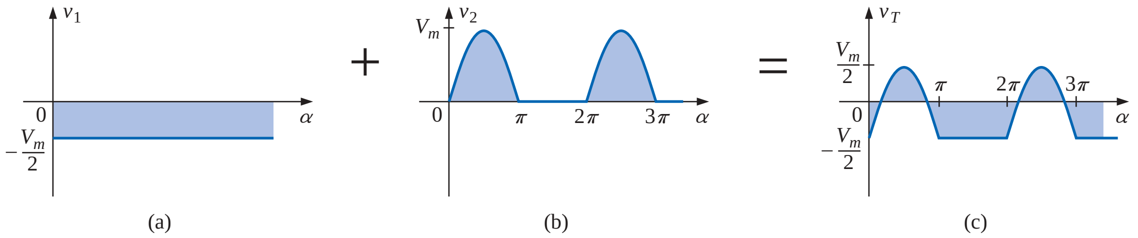


FIG. 25.19

Lowering a waveform with the addition of a negative dc component.

If either waveform were shifted to the right or left, the phase shift would be subtracted from or added to, respectively, the sine and cosine terms. The dc term would not change with a shift to the right or left.

If the half-wave rectified signal is shifted 90° to the left, as in Fig. 25.20, the Fourier series becomes

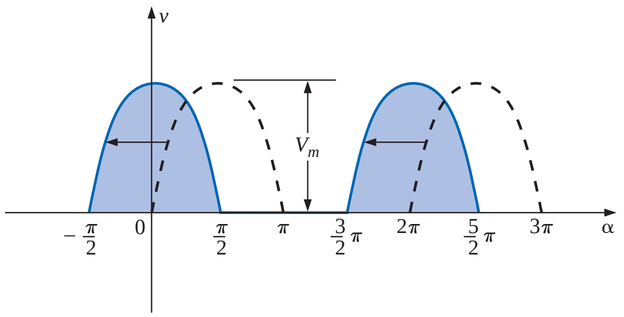


FIG. 25.20

Changing the phase angle of a waveform.

$$v = 0.318V_m + 0.500V_m \underbrace{\sin(\alpha + 90°)}_{\cos\alpha} - 0.212V_m \cos 2(\alpha + 90°) - 0.0424V_m \cos 4(\alpha + 90°) + \cdots$$

$$= 0.318V_m + 0.500V_m \cos\alpha - 0.212V_m \cos(2\alpha + 180°) - 0.0424V_m \cos(4\alpha + 360°) + \cdots$$

and $$v = 0.318V_m + 0.500V_m \cos\alpha + 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha + \cdots$$

25.3 CIRCUIT RESPONSE TO A NONSINUSOIDAL INPUT

The Fourier series representation of a nonsinusoidal input can be applied to a linear network using the principle of superposition. Recall that this theorem allowed us to consider the effects of each source of a circuit independently. If we replace the nonsinusoidal input with the terms of the Fourier series deemed necessary for practical considerations, we can use superposition to find the response of the network to each term (Fig. 25.21).

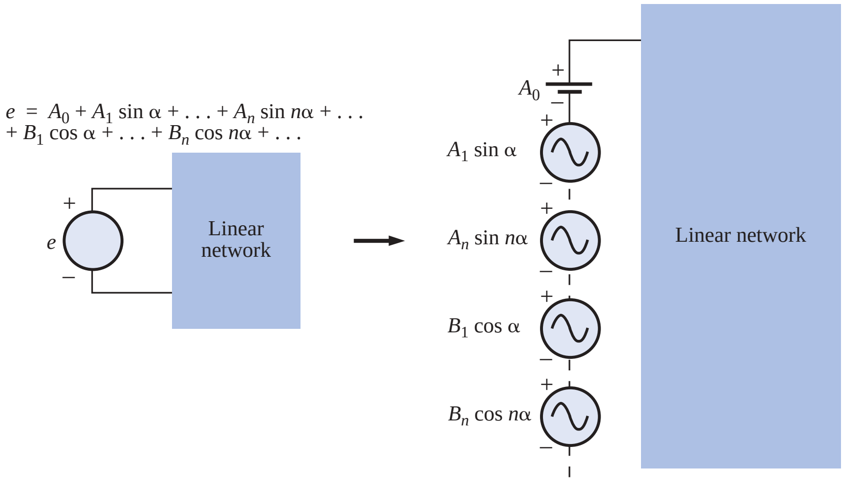


FIG. 25.21
Setting up the application of a Fourier series of terms to a linear network.

The total response of the system is then the algebraic sum of the values obtained for each term. The major change between using this theorem for nonsinusoidal circuits and using it for the circuits previously described is that the frequency will be different for each term in the nonsinusoidal application. Therefore, the reactances

$$X_L = 2\pi fL \quad \text{and} \quad X_C = \frac{1}{2\pi fC}$$

will change for each term of the input voltage or current.

In Chapter 13, we found that the rms value of any waveform was given by

$$\sqrt{\frac{1}{T}\int_0^T f^2(t)\,dt}$$

If we apply this equation to the following Fourier series:

$$v(\alpha) = V_0 + V_{m_1}\sin\alpha + \cdots + V_{m_n}\sin n\alpha + V'_{m_1}\cos\alpha + \cdots + V'_{m_n}\cos n\alpha$$

then

$$\boxed{V_{\text{rms}} = \sqrt{V_0^2 + \frac{V_{m_1}^2 + \cdots + V_{m_n}^2 + V'^2_{m_1} + \cdots + V'^2_{m_n}}{2}}} \tag{25.11}$$

However, since

$$\frac{V_{m_1}^2}{2} = \left(\frac{V_{m_1}}{\sqrt{2}}\right)\left(\frac{V_{m_1}}{\sqrt{2}}\right) = (V_{1_{\text{rms}}})(V_{1_{\text{rms}}}) = V_{1_{\text{rms}}}^2$$

then

$$\boxed{V_{\text{rms}} = \sqrt{V_0^2 + V_{1_{\text{rms}}}^2 + \cdots + V_{n_{\text{rms}}}^2 + V'^2_{1_{\text{rms}}} + \cdots + V'^2_{n_{\text{rms}}}}} \tag{25.12}$$

Similarly, for

$$i(\alpha) = I_0 + I_{m_1}\sin\alpha + \cdots + I_{m_n}\sin n\alpha + I'_{m_1}\cos\alpha + \cdots + I'_{m_n}\cos n\alpha$$

we have

$$\boxed{I_{\text{rms}} = \sqrt{I_0^2 + \frac{I_{m_1}^2 + \cdots + I_{m_n}^2 + I'^2_{m_1} + \cdots + I'^2_{m_n}}{2}}} \tag{25.13}$$

and

$$\boxed{I_{\text{rms}} = \sqrt{I_0^2 + I_{1_{\text{rms}}}^2 + \cdots + I_{n_{\text{rms}}}^2 + I'^2_{1_{\text{rms}}} + \cdots + I'^2_{n_{\text{rms}}}}} \tag{25.14}$$

The total power delivered is the sum of that delivered by the corresponding terms of the voltage and current. In the following equations, all voltages and currents are rms values:

$$\boxed{P_T = V_0 I_0 + V_1 I_1 \cos\theta_1 + \cdots + V_n I_n \cos\theta_n + \cdots} \tag{25.15}$$

$$\boxed{P_T = I_0^2 R + I_1^2 R + \cdots + I_n^2 R + \cdots} \tag{25.16}$$

or

$$\boxed{P_T = I_{\text{rms}}^2 R} \tag{25.17}$$

with I_{rms} as defined by Eq. (25.13), and, similarly,

$$\boxed{P_T = \frac{V_{\text{rms}}^2}{R}} \tag{25.18}$$

with V_{rms} as defined by Eq. (25.11).

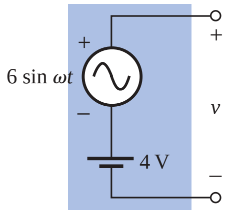


FIG. 25.22
Example 25.5.

EXAMPLE 25.5

a. Sketch the input resulting from the combination of sources in Fig. 25.22.

b. Determine the rms value of the input of Fig. 25.22.

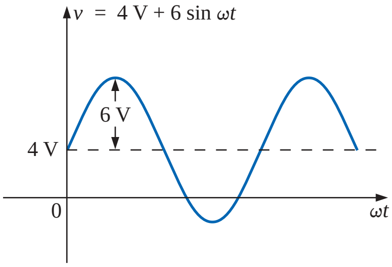


FIG. 25.23
Wave pattern generated by the source of Fig. 25.22.

Solutions:

a. Note Fig. 25.23.

b. Eq. (25.12):

$$V_{\text{rms}} = \sqrt{V_0^2 + \frac{V_m^2}{2}}$$

$$= \sqrt{(4\text{ V})^2 + \frac{(6\text{ V})^2}{2}} = \sqrt{16 + \frac{36}{2}}\text{ V} = \sqrt{34}\text{ V}$$

$$= \mathbf{5.831\ V}$$

It is particularly interesting to note from Example 25.5 that the rms value of a waveform having both dc and ac components is not simply the sum of the effective values of each. In other words, there is a temptation in the absence of Eq. (25.12) to state that $V_{\text{rms}} = 4\text{ V} + 0.707(6\text{ V}) = 8.242\text{ V}$, which is incorrect and, in fact, exceeds the correct level by some 41%.

Instrumentation

It is important to realize that not every DMM will read the rms value of nonsinusoidal waveforms such as the one appearing in Fig. 25.23. Many are designed to read the rms value of sinusoidal waveforms only. It is important to read the manual provided with the meter to see if it is a *true rms* meter that can read the rms value of any waveform.

We learned in Chapter 13 that the rms value of a square wave is the peak value of the waveform. Let us test this result using the Fourier expansion and Eq. (25.11).

EXAMPLE 25.6 Determine the rms value of the square wave of Fig. 25.15 with $V_m = 20$ V using the first six terms of the Fourier expansion, and compare the result to the actual rms value of 20 V.

Solution:

$$v = \frac{4}{\pi}(20\text{ V})\sin\omega t + \frac{4}{\pi}\left(\frac{1}{3}\right)(20\text{ V})\sin 3\omega t + \frac{4}{\pi}\left(\frac{1}{5}\right)(20\text{ V})\sin 5\omega t + \frac{4}{\pi}\left(\frac{1}{7}\right)(20\text{ V})\sin 7\omega t$$

$$+ \frac{4}{\pi}\left(\frac{1}{9}\right)(20\text{ V})\sin 9\omega t + \frac{4}{\pi}\left(\frac{1}{11}\right)(20\text{ V})\sin 11\omega t$$

$$v = 25.465\sin\omega t + 8.488\sin 3\omega t + 5.093\sin 5\omega t + 3.638\sin 7\omega t + 2.829\sin 9\omega t + 2.315\sin 11\,\omega t$$

Eq. (25.11):

$$V_{\text{rms}} = \sqrt{V_0^2 + \frac{V_{m_1}^2 + V_{m_2}^2 + V_{m_3}^2 + V_{m_4}^2 + V_{m_5}^2 + V_{m_6}^2}{2}}$$

$$= \sqrt{(0\text{ V})^2 + \frac{(25.465\text{ V})^2 + (8.488\text{ V})^2 + (5.093\text{ V})^2 + (3.638\text{ V})^2 + (2.829\text{ V})^2 + (2.315\text{ V})^2}{2}}$$

$$= \mathbf{19.66\ V}$$

The solution differs less than 0.4 V from the correct answer of 20 V. However, each additional term in the Fourier series will bring the result closer to the 20-V level. An infinite number would result in an exact solution of 20 V.

EXAMPLE 25.7 The input to the circuit of Fig. 25.24 is the following:

$$e = 12 + 10 \sin 2t$$

a. Find the current i and the voltages v_R and v_C.
b. Find the rms values of i, v_R, and v_C.
c. Find the power delivered to the circuit.

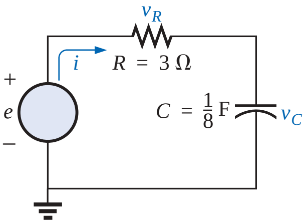


FIG. 25.24
Example 25.7.

Solutions:

a. Redraw the original circuit as shown in Fig. 25.25. Then apply superposition:

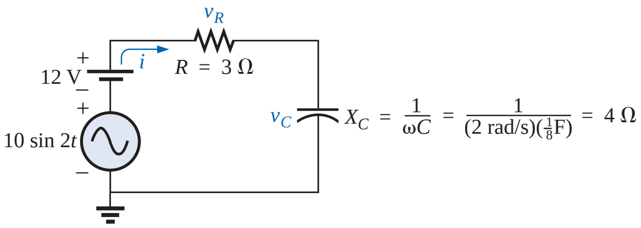


FIG. 25.25
Circuit of Fig. 25.24 with the components of the Fourier series input.

1. *For the 12-V dc supply portion of the input*, $I = 0$ since the capacitor is an open circuit to dc when v_C has reached its final (steady-state) value. Therefore,

$$V_R = IR = 0\text{ V} \quad \text{and} \quad V_C = 12\text{ V}$$

2. *For the ac supply,*

$$\mathbf{Z} = 3\,\Omega - j\,4\,\Omega = 5\,\Omega\,\angle{-53.13^\circ}$$

$$\text{and}\quad \mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}} = \frac{\frac{10}{\sqrt{2}}\text{ V}\,\angle 0^\circ}{5\,\Omega\,\angle{-53.13^\circ}} = \frac{2}{\sqrt{2}}\text{ A}\,\angle{+53.13^\circ}$$

$$\mathbf{V}_R = (I\,\angle\theta)(R\,\angle 0^\circ) = \left(\frac{2}{\sqrt{2}}\text{ A}\,\angle{+53.13^\circ}\right)(3\,\Omega\,\angle 0^\circ)$$

$$= \frac{6}{\sqrt{2}}\text{ V}\,\angle{+53.13^\circ}$$

and

$$\mathbf{V}_C = (I\,\angle\theta)(X_C\,\angle{-90^\circ}) = \left(\frac{2}{\sqrt{2}}\text{ A}\,\angle{+53.13^\circ}\right)(4\,\Omega\,\angle{-90^\circ})$$

$$= \frac{8}{\sqrt{2}}\text{ V}\,\angle{-36.87^\circ}$$

In the time domain,

$$i = \mathbf{0 + 2\sin(2}\boldsymbol{t}\mathbf{ + 53.13^\circ)}$$

Note that even though the dc term was present in the expression for the input voltage, the dc term for the current in this circuit is zero:

$$v_R = \mathbf{0 + 6\sin(2}\boldsymbol{t}\mathbf{ + 53.13^\circ)}$$

$$\text{and}\qquad v_C = \mathbf{12 + 8\sin(2}\boldsymbol{t}\mathbf{ - 36.87^\circ)}$$

b. Eq. (25.14): $I_{\text{rms}} = \sqrt{(0)^2 + \dfrac{(2\text{ A})^2}{2}} = \sqrt{2}\text{ V} = \mathbf{1.414\text{ A}}$

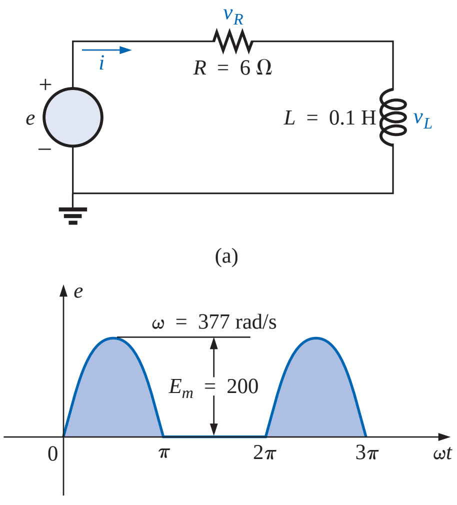
Eq. (25.12): $V_{R_{\text{rms}}} = \sqrt{(0)^2 + \dfrac{(6\text{ V})^2}{2}} = \sqrt{18}\text{ V} = \mathbf{4.243\text{ V}}$

Eq. (25.12): $V_{C_{\text{rms}}} = \sqrt{(12\text{ V})^2 + \dfrac{(8\text{ V})^2}{2}} = \sqrt{176}\text{ V} = \mathbf{13.267\text{ V}}$

c. $P = I_{\text{rms}}^2 R = \left(\dfrac{2}{\sqrt{2}}\text{ A}\right)^2 (3\ \Omega) = \mathbf{6\text{ W}}$

EXAMPLE 25.8 Find the response of the circuit of Fig. 25.26 to the input shown.

$$e = 0.318E_m + 0.500E_m \sin \omega t - 0.212E_m \cos 2\omega t - 0.0424E_m \cos 4\omega t + \cdots$$



(a)

(b)

FIG. 25.26
Example 25.8.

Solution: For discussion purposes, only the first three terms will be used to represent *e*. Converting the cosine terms to sine terms and substituting for E_m gives us

$$e = 63.60 + 100.0 \sin \omega t - 42.40 \sin(2\omega t + 90°)$$

Using phasor notation, the original circuit becomes like the one shown in Fig. 25.27.

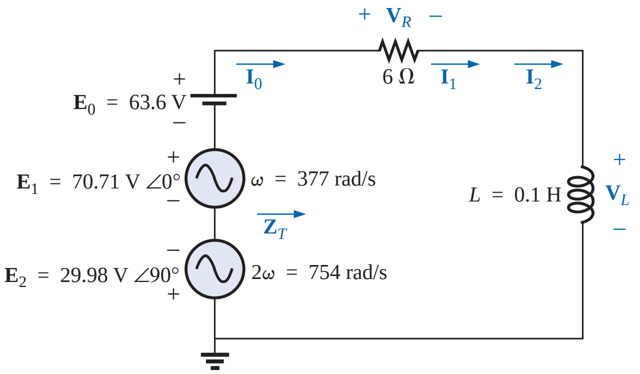


FIG. 25.27
Circuit of Fig. 25.26 with the components of the Fourier series input.

Applying Superposition *For the dc term* ($E_0 = 63.6$ V):

$$X_L = 0 \qquad \text{(short for dc)}$$

$$\mathbf{Z}_T = R\ \angle 0° = 6\ \Omega\ \angle 0°$$

$$I_0 = \frac{E_0}{R} = \frac{63.6\text{ V}}{6\ \Omega} = 10.60\text{ A}$$

$$V_{R_0} = I_0 R = E_0 = 63.60\text{ V}$$

$$V_{L_0} = 0$$

The average power is

$$P_0 = I_0^2 R = (10.60\text{ A})^2(6\ \Omega) = 674.2\text{ W}$$

For the fundamental term ($\mathbf{E}_1 = 70.71\text{ V}\ \angle 0°$, $\omega = 377$):

$$X_{L_1} = \omega L = (377\text{ rad/s})(0.1\text{ H}) = 37.7\ \Omega$$

$$\begin{aligned}
\mathbf{Z}_{T_1} &= 6\ \Omega + j\,37.7\ \Omega = 38.17\ \Omega\ \angle 80.96^\circ \\
\mathbf{I}_1 &= \frac{\mathbf{E}_1}{\mathbf{Z}_{T_1}} = \frac{70.71\ \text{V}\ \angle 0^\circ}{38.17\ \Omega\ \angle 80.96^\circ} = 1.85\ \text{A}\ \angle -80.96^\circ \\
\mathbf{V}_{R_1} &= (I_1\ \angle\theta)(R\ \angle 0^\circ) = (1.85\ \text{A}\ \angle -80.96^\circ)(6\ \Omega\ \angle 0^\circ) \\
&= 11.10\ \text{V}\ \angle -80.96^\circ \\
\mathbf{V}_{L_1} &= (I_1\ \angle\theta)(X_{L_1}\ \angle 90^\circ) = (1.85\ \text{A}\ \angle -80.96^\circ)(37.7\ \Omega\ \angle 90^\circ) \\
&= 69.75\ \text{V}\ \angle 9.04^\circ
\end{aligned}$$

The average power is

$$P_1 = I_1^2 R = (1.85\ \text{A})^2(6\ \Omega) = 20.54\ \text{W}$$

For the second harmonic ($\mathbf{E}_2 = 29.98\ \text{V}\ \angle -90^\circ$, $\omega = 754$): The phase angle of $\mathbf{E}_2$ was changed to -90° to give it the same polarity as the input voltages $\mathbf{E}_0$ and $\mathbf{E}_1$.

$$\begin{aligned}
X_{L_2} &= \omega L = (754\ \text{rad/s})(0.1\ \text{H}) = 75.4\ \Omega \\
\mathbf{Z}_{T_2} &= 6\ \Omega + j\,75.4\ \Omega = 75.64\ \Omega\ \angle 85.45^\circ \\
\mathbf{I}_2 &= \frac{\mathbf{E}_2}{\mathbf{Z}_{T_2}} = \frac{29.98\ \text{V}\ \angle -90^\circ}{75.64\ \Omega\ \angle 85.45} = 0.396\ \text{A}\ \angle -174.45^\circ \\
\mathbf{V}_{R_2} &= (I_2\ \angle\theta)(R\ \angle 0^\circ) = (0.396\ \text{A}\ \angle -174.45^\circ)(6\ \Omega\ \angle 0^\circ) \\
&= 2.38\ \text{V}\ \angle -174.45^\circ \\
\mathbf{V}_{L_2} &= (I_2\ \angle\theta)(X_{L_2}\ \angle 90^\circ) = (0.396\ \text{A}\ \angle -174.45^\circ)(75.4\ \Omega\ \angle 90^\circ) \\
&= 29.9\ \text{V}\ \angle -84.45^\circ
\end{aligned}$$

The average power is

$$P_2 = I_2^2 R = (0.396\ \text{A})^2(6\ \Omega) = 0.941\ \text{W}$$

The Fourier series expansion for i is

$$i = \mathbf{10.6 + \sqrt{2}(1.85)\ sin(377\mathit{t} - 80.96^\circ) + \sqrt{2}(0.396)\ sin(754\mathit{t} - 174.45^\circ)}$$

and

$$I_{\text{rms}} = \sqrt{(10.6\ \text{A})^2 + (1.85\ \text{A})^2 + (0.396\ \text{A})^2} = \mathbf{10.77\ A}$$

The Fourier series expansion for v_R is

$$v_R = \mathbf{63.6 + \sqrt{2}(11.10)\ sin(377\mathit{t} - 80.96^\circ) + \sqrt{2}(2.38)\ sin(754\mathit{t} - 174.45^\circ)}$$

and

$$V_{R_{\text{rms}}} = \sqrt{(63.6\ \text{V})^2 + (11.10\ \text{V})^2 + (2.38\ \text{V})^2} = \mathbf{64.61\ V}$$

The Fourier series expansion for v_L is

$$v_L = \mathbf{\sqrt{2}(69.75)\ sin(377\mathit{t} + 9.04^\circ) + \sqrt{2}(29.93)\ sin(754\mathit{t} - 84.45^\circ)}$$

and $$V_{L_{\text{rms}}} = \sqrt{(69.75\ \text{V})^2 + (29.93\ \text{V})^2} = \mathbf{75.90\ V}$$

The total average power is

$$P_T = I_{\text{rms}}^2 R = (10.77\ \text{A})^2(6\ \Omega) = \mathbf{695.96\ W} = P_0 + P_1 + P_2$$

25.4 ADDITION AND SUBTRACTION OF NONSINUSOIDAL WAVEFORMS

The Fourier series expression for the waveform resulting from the addition or subtraction of two nonsinusoidal waveforms can be found using phasor algebra if the terms having the same frequency are considered separately.

For example, the sum of the following two nonsinusoidal waveforms is found using this method:

$$v_1 = 30 + 20 \sin 20t + \cdots + 5 \sin(60t + 30^\circ)$$
$$v_2 = 60 + 30 \sin 20t + 20 \sin 40t + 10 \cos 60t$$

1. dc terms:

$$V_{T_0} = 30\text{ V} + 60\text{ V} = 90\text{ V}$$

2. $\omega = 20$:

$$V_{T_{1(\max)}} = 30\text{ V} + 20\text{ V} = 50\text{ V}$$

and $$v_{T_1} = 50 \sin 20t$$

3. $\omega = 40$:

$$v_{T_2} = 20 \sin 40t$$

4. $\omega = 60$:

$$5 \sin(60t + 30^\circ) = (0.707)(5)\text{ V} \angle 30^\circ = 3.54\text{ V} \angle 30^\circ$$
$$\begin{aligned} 10 \cos 60t &= 10 \sin(60t + 90^\circ) \Rightarrow (0.707)(10)\text{ V} \angle 90^\circ \\ &= 7.07\text{ V} \angle 90^\circ \end{aligned}$$
$$\begin{aligned} \mathbf{V}_{T_3} &= 3.54\text{ V} \angle 30^\circ + 7.07\text{ V} \angle 90^\circ \\ &= 3.07\text{ V} + j\,1.77\text{ V} + j\,7.07\text{ V} = 3.07\text{ V} + j\,8.84\text{ V} \\ \mathbf{V}_{T_3} &= 9.36\text{ V} \angle 70.85^\circ \end{aligned}$$

and $$v_{T_3} = 13.24 \sin(60t + 70.85^\circ)$$

with

$$v_T = v_1 + v_2 = \mathbf{90 + 50 \sin 20}\boldsymbol{t}\mathbf{ + 20 \sin 40}\boldsymbol{t}\mathbf{ + 13.24 \sin(60}\boldsymbol{t}\mathbf{ + 70.85^\circ)}$$

25.5 COMPUTER ANALYSIS

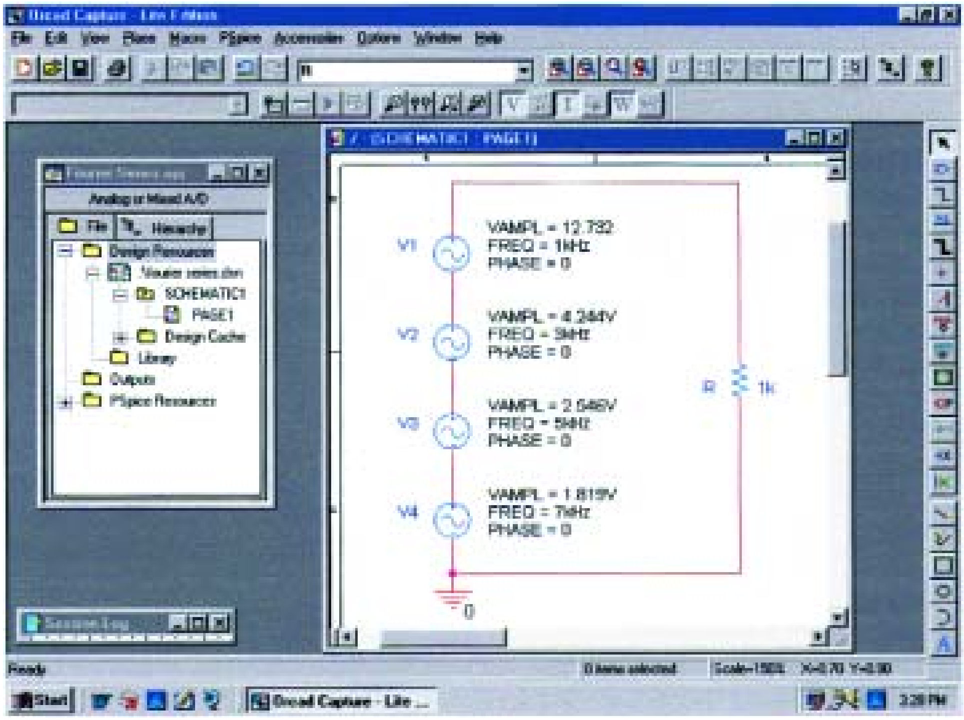
PSpice

Fourier Series The computer analysis will begin with a verification of the waveform of Fig. 25.17, demonstrating that only four terms of a Fourier series can generate a waveform that has a number of characteristics of a square wave. The square wave has a peak value of 10 V at a frequency of 1 kHz, resulting in the following Fourier series using Eq. (25.9) (and recognizing that $\omega = 2\pi f = 6283.19$ rad/s):

$$\begin{aligned} v &= \frac{4}{\pi}(10\text{ V})\left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \frac{1}{7} \sin 7\omega t\right) \\ &= 12.732 \sin \omega t + 4.244 \sin 3\omega t + 2.546 \sin 5\omega t + 1.819 \sin 7\omega t \end{aligned}$$

Each term of the Fourier series is treated as an independent ac source as shown in Fig. 25.28 with its peak value and applicable frequency. The sum of the source voltages will appear across the resistor *R* and will generate the waveform of Fig. 25.29.

Each source used **VSIN**, and since we wanted to display the result against time, we chose **Time Domain(Transient)** in the **Simulation Settings.** For each source the **Property Editor** dialog box was called up, and **AC, FREQ, PHASE, VAMPL,** and **VOFF** (at 0 V) were set, although due to limited space only **VAMPL, FREQ,** and **PHASE** were



V1
VAMPL = 12.732
FREQ = 1kHz
PHASE = 0

V2
VAMPL = 4.244V
FREQ = 3kHz
PHASE = 0

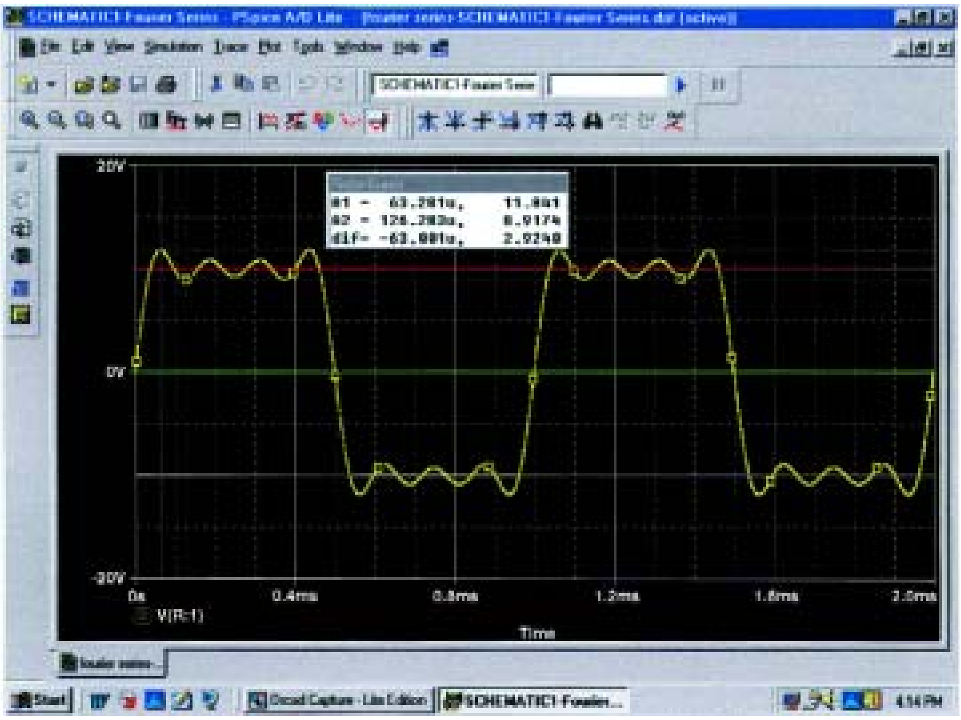
V3
VAMPL = 2.546V
FREQ = 5kHz
PHASE = 0

V4
VAMPL = 1.819V
FREQ = 7kHz
PHASE = 0

R 1k

FIG. 25.28

Using PSpice to apply four terms of the Fourier expansion of a 10-V square wave to a load resistor of 1 kΩ.



```
A1 =   63.201u,       11.001
A2 =  126.283u,       8.9174
dif= -63.081u,        2.0840
```

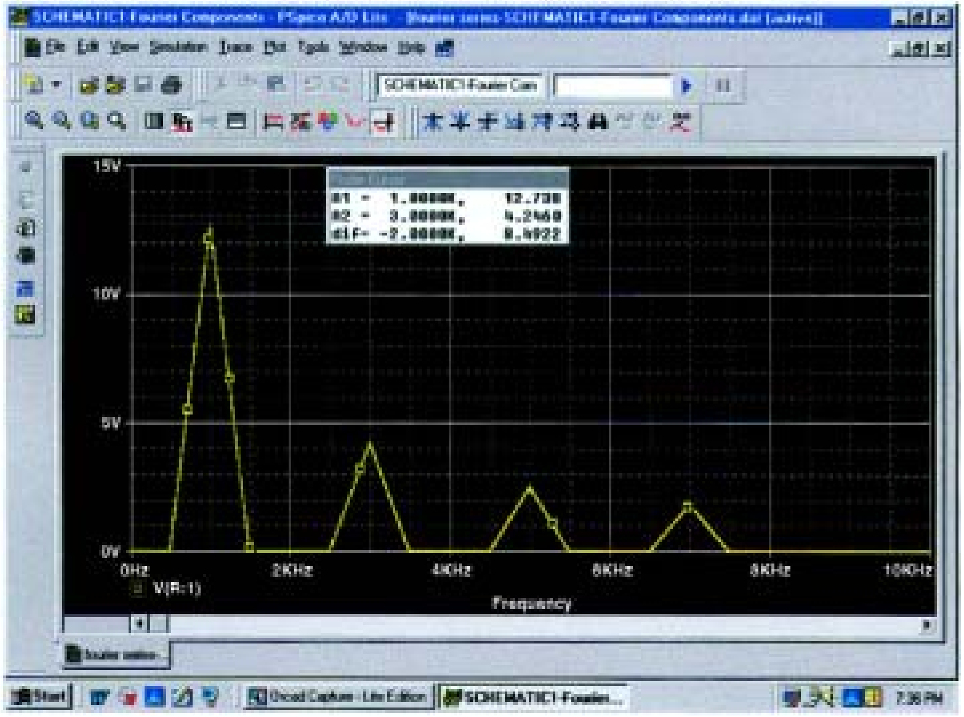
FIG. 25.29

The resulting waveform of the voltage across the resistor R of Fig. 25.28.

displayed in Fig. 25.28. Under **Display** the remaining quantitites were all set on **Do Not Display.**

The **Run to time** was set at 2 ms so that two cycles of the fundamental frequency of 1 kHz would appear. The **Start saving data after** will remain at the default value of 0 s, and the **Maximum step size** will be 1 μs, even though 2 ms/1000 = 2 μs, because we want to have additional plot points for the complex waveform. Once the **SCHEMATIC1** window appears, **Trace-Add Trace-V(R:1)-OK** will result in the waveform of Fig. 25.29. The horizontal line at 0 V was made heavier by right-clicking on the line, selecting **Properties,** and then choosing the green color and wider line. Click **OK,** and the wider line of Fig. 25.29 will result, making it a great deal clearer where the 0-V line is located. Through the same process the curve was made yellow and wider as shown in the same figure. Using the cursors, we find that the first peak will reach 11.84 V and then drop to 8.920 V. The average value of the waveform is clearly +10 V in the positive region as shown by the red line entered using **Plot-Label-Line.** In every respect the waveform is certainly beginning to have the characteristics of a periodic square wave with a peak value of 10 V and a frequency of 1 kHz.

Fourier Components A frequency spectrum plot revealing the magnitude and frequency of each component of a Fourier series can be obtained by returning to **Plot** and selecting **Axis Settings** followed by **X Axis** and then **Fourier** under **Processing Options.** Click **OK,** and a number of spikes will appear on the far left of the screen, with a frequency spectrum that extends from 0 Hz to 600 kHz. By selecting **Plot-Axis Settings** again, going to **Data Range,** and selecting **User Defined,**



SCHEMATIC1-Fourier Components - PSpice A/D Lite - [fourier series-SCHEMATIC1-Fourier Components.dat (active)]

```
A1 =   1.0000K,      12.738
A2 =   3.0000K,      4.2460
dif= -2.0000K,      8.4922
```

FIG. 25.30
The Fourier components of the waveform of Fig. 25.29.

you can change the range to 0 Hz to 10 kHz since this is the range of interest for this waveform. Click **OK,** and the graph of Fig. 25.30 will result, giving the magnitude and frequency of the components of the waveform. Using the left cursor, we find that the highest peak is 12.738 V at 1 kHz, comparing very well with the source **V1** having a peak value of 12.732 V at 1 kHz. Using the right-click cursor, we can move over to 3 kHz and find a magnitude of 4.246 V, again comparing very well with source **V2** with a peak value of 4.244 V.

PROBLEMS

SECTION 25.2 **Fourier Series**

1. For the waveforms of Fig. 25.31, determine whether the following will be present in the Fourier series representation:
    - **a.** dc term
    - **b.** cosine terms
    - **c.** sine terms
    - **d.** even-ordered harmonics
    - **e.** odd-ordered harmonics

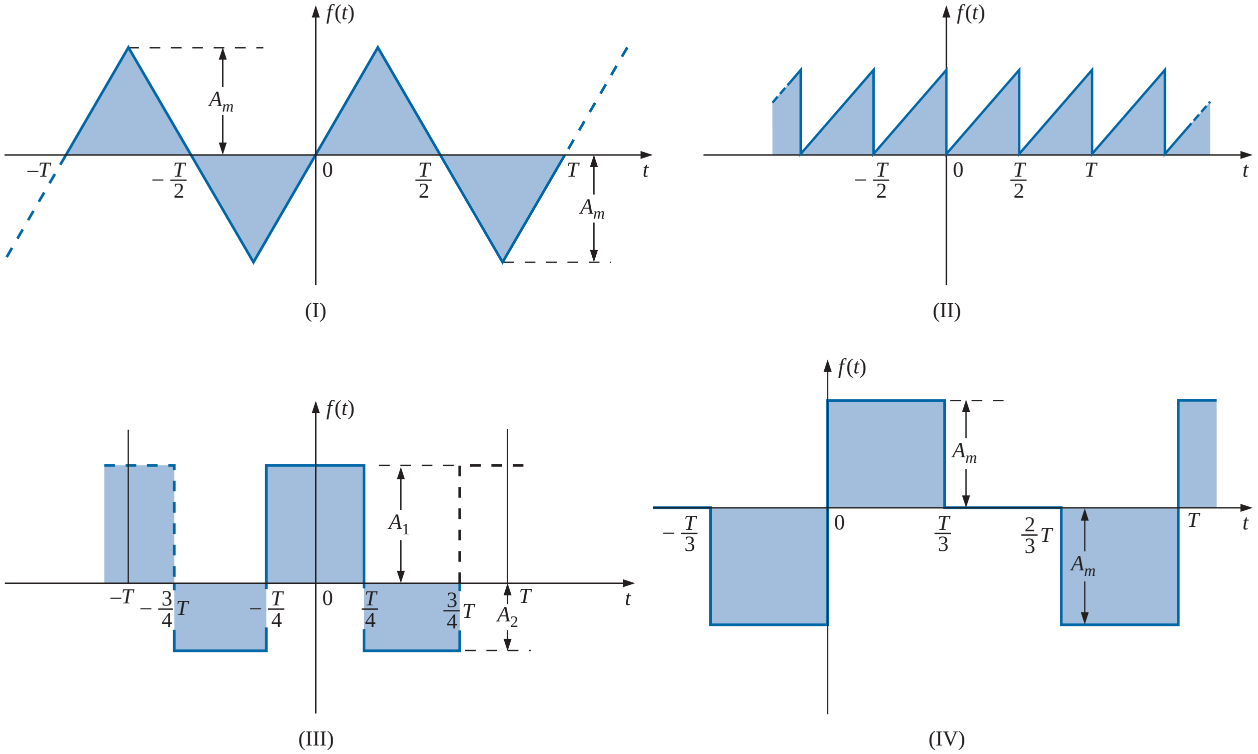


FIG. 25.31
Problem 1.

2. If the Fourier series for the waveform of Fig. 25.32(a) is

$$i = \frac{2I_m}{\pi}\left(1 + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{35}\cos 6\omega t + \cdots\right)$$

find the Fourier series representation for waveforms (b) through (d).

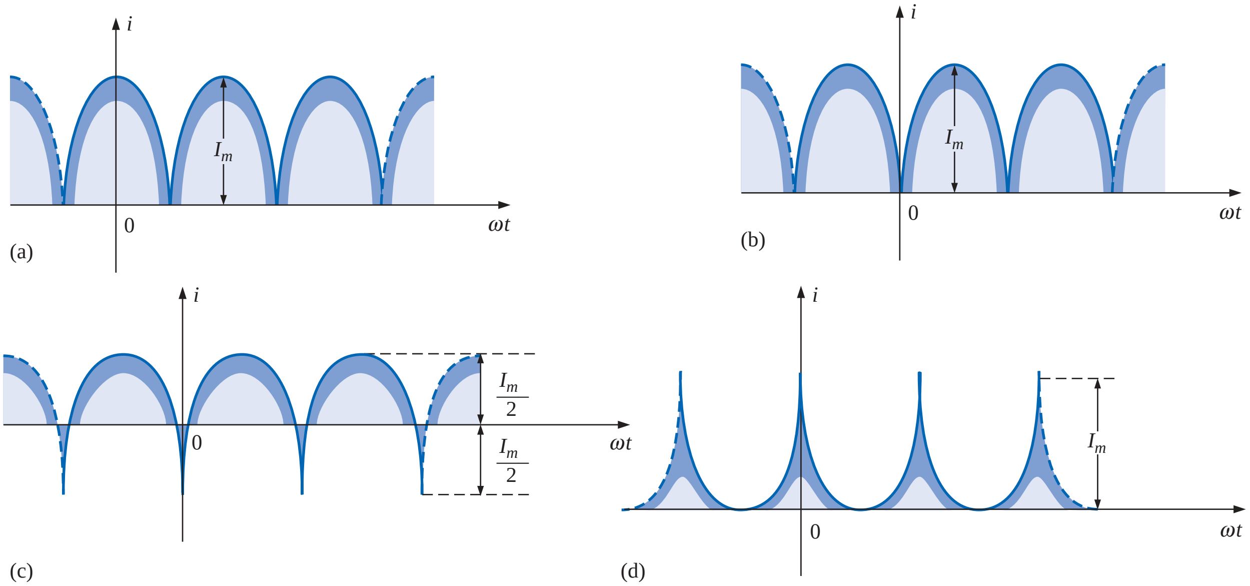


FIG. 25.32
Problem 2.

3. Sketch the following nonsinusoidal waveforms with $\alpha = \omega t$ as the abscissa:
- **a.** $v = -4 + 2\sin\alpha$
- **b.** $v = (\sin\alpha)^2$
- **c.** $i = 2 - 2\cos\alpha$

4. Sketch the following nonsinusoidal waveforms with α as the abscissa:
- **a.** $i = 3\sin\alpha - 6\sin 2\alpha$
- **b.** $v = 2\cos 2\alpha + \sin\alpha$

5. Sketch the following nonsinusoidal waveforms with ωt as the abscissa:
- **a.** $i = 50\sin\omega t + 25\sin 3\omega t$
- **b.** $i = 50\sin\alpha - 25\sin 3\alpha$
- **c.** $i = 4 + 3\sin\omega t + 2\sin 2\omega t - 1\sin 3\omega t$

SECTION 25.3 Circuit Response to a Nonsinusoidal Input

6. Find the average and effective values of the following nonsinusoidal waves:
- **a.** $v = 100 + 50\sin\omega t + 25\sin 2\omega t$
- **b.** $i = 3 + 2\sin(\omega t - 53^\circ) + 0.8\sin(2\omega t - 70^\circ)$

7. Find the rms value of the following nonsinusoidal waves:
- **a.** $v = 20\sin\omega t + 15\sin 2\omega t - 10\sin 3\omega t$
- **b.** $i = 6\sin(\omega t + 20^\circ) + 2\sin(2\omega t + 30^\circ) - 1\sin(3\omega t + 60^\circ)$

8. Find the total average power to a circuit whose voltage and current are as indicated in Problem 6.

9. Find the total average power to a circuit whose voltage and current are as indicated in Problem 7.

10. The Fourier series representation for the input voltage to the circuit of Fig. 25.33 is

$$e = 18 + 30 \sin 400t$$

- **a.** Find the nonsinusoidal expression for the current i.
- **b.** Calculate the rms value of the current.
- **c.** Find the expression for the voltage across the resistor.
- **d.** Calculate the rms value of the voltage across the resistor.
- **e.** Find the expression for the voltage across the reactive element.
- **f.** Calculate the rms value of the voltage across the reactive element.
- **g.** Find the average power delivered to the resistor.

11. Repeat Problem 10 for

$$e = 24 + 30 \sin 400t + 10 \sin 800t$$

12. Repeat Problem 10 for the following input voltage:

$$e = -60 + 20 \sin 300t - 10 \sin 600t$$

13. Repeat Problem 10 for the circuit of Fig. 25.34.

***14.** The input voltage of Fig. 25.35(a) to the circuit of Fig. 25.35(b) is a full-wave rectified signal having the following Fourier series expansion:

$$e = \frac{(2)(100\text{ V})}{\pi}\left(1 + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{53}\cos 6\omega t + \cdots\right)$$

where $\omega = 377$.

- **a.** Find the Fourier series expression for the voltage v_o using only the first three terms of the expression.
- **b.** Find the rms value of v_o.
- **c.** Find the average power delivered to the 1-kΩ resistor.

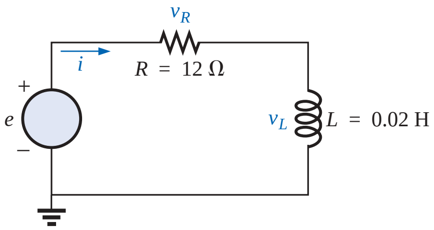


FIG. 25.33
Problems 10, 11, and 12.

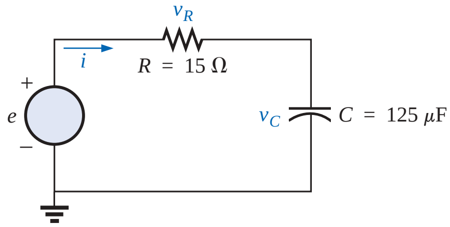
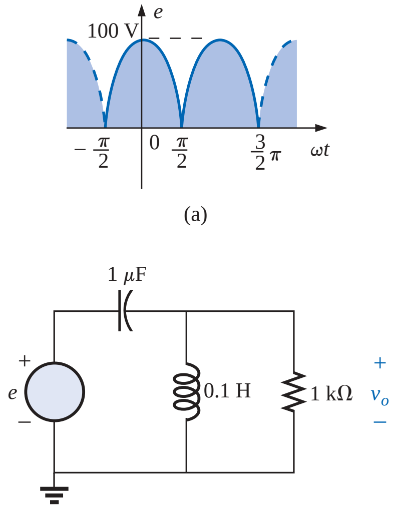


FIG. 25.34
Problem 13.

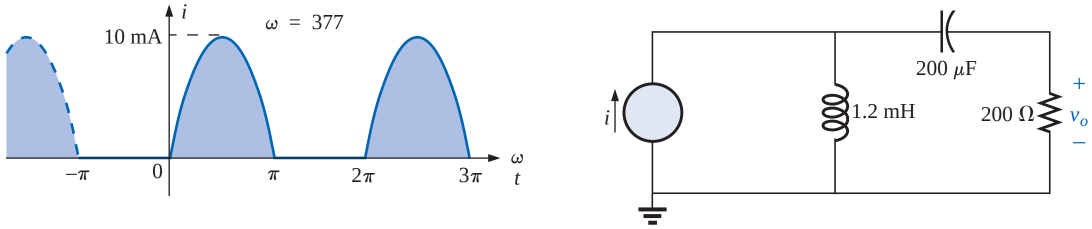


(a)

(b)

FIG. 25.35
Problem 14.

***15.** Find the Fourier series expression for the voltage v_o of Fig. 25.36.



(a)

(b)

FIG. 25.36
Problem 15.

SECTION 25.4 Addition and Subtraction of Nonsinusoidal Waveforms

16. Perform the indicated operations on the following nonsinusoidal waveforms:

a.

$[60 + 70 \sin \omega t + 20 \sin(2\omega t + 90°) + 10 \sin(3\omega t + 60°)]$
$+ [20 + 30 \sin \omega t - 20 \cos 2\omega t + 5 \cos 3\omega t]$

b.

$[20 + 60 \sin \alpha + 10 \sin(2\alpha - 180°) + 5 \cos(3\alpha + 90°)]$
$- [5 - 10 \sin \alpha + 4 \sin(3\alpha - 30°)]$

17. Find the nonsinusoidal expression for the current i_s of the diagram of Fig. 25.37.

$i_2 = 10 + 30 \sin 20t - 0.5 \sin(40t + 90°)$

$i_1 = 20 + 4 \sin(20t + 90°) + 0.5 \sin(40t + 30°)$

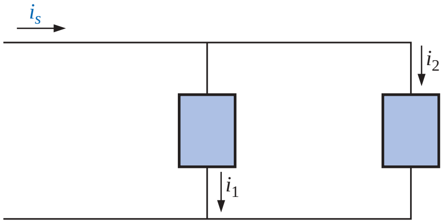


FIG. 25.37
Problem 17.

18. Find the nonsinusoidal expression for the voltage e of the diagram of Fig. 25.38.

$v_1 = 20 - 200 \sin 600t + 100 \cos 1200t + 75 \sin 1800t$

$v_2 = -10 + 150 \sin(600t + 30°) + 50 \sin(1800t + 60°)$

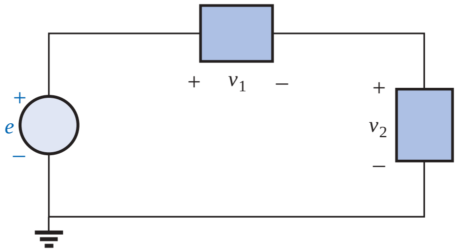


FIG. 25.38
Problem 18.

SECTION 25.5 Computer Analysis

PSpice or Electronics Workbench

19. Plot the waveform of Fig. 25.13 for two or three cycles. Then obtain the Fourier components, and compare them to the applied signal.

20. Plot a half-rectified waveform with a peak value of 20 V using Eq. (25.10). Use the dc term, the fundamental term, and four harmonics. Compare the resulting waveform to the ideal half-rectified waveform.

21. Demonstrate the effect of adding two more terms to the waveform of Fig. 25.29, and generate the Fourier spectrum.

Computer Language (C++, QBASIC, Pascal, etc.)

22. Write a program to obtain the Fourier expansion resulting from the addition of two nonsinusoidal waveforms.

23. Write a program to determine the sum of the first 10 terms of Eq. (25.9) at $\omega t = \pi/2, \pi,$ and $(3/2)\pi$, and compare your results to the values determined by Fig. 25.15. That is, enter Eq. (25.9) into memory, and calculate the sum of the terms at the points listed above.

24. Given any nonsinusoidal function, write a program that will determine the average and rms values of the waveform. The program should request the data required from the nonsinusoidal function.

25. Write a program that will provide a general solution for the network of Fig. 25.24 for a single dc and ac term in the applied voltage. In other words, the parameter values are given along with the particulars regarding the applied signal, and the nonsinusoidal expression for the current and each voltage is generated by the program.

GLOSSARY

Axis symmetry A sinusoidal or nonsinusoidal function that has symmetry about the vertical axis.

Even harmonics The terms of the Fourier series expansion that have frequencies that are even multiples of the fundamental component.

Fourier series A series of terms, developed in 1826 by Baron Jean Fourier, that can be used to represent a nonsinusoidal function.

Fundamental component The minimum frequency term required to represent a particular waveform in the Fourier series expansion.

Half-wave (mirror) symmetry A sinusoidal or nonsinusoidal function that satisfies the relationship

$$f(t) = -f\left(\frac{T}{2} + t\right)$$

Harmonic terms The terms of the Fourier series expansion that have frequencies that are integer multiples of the fundamental component.

Nonsinusoidal waveform Any waveform that differs from the fundamental sinusoidal function.

Odd harmonics The terms of the Fourier series expansion that have frequencies that are odd multiples of the fundamental component.

Point symmetry A sinusoidal or nonsinusoidal function that satisfies the relationship $f(\alpha) = -f(-\alpha)$.