PROBLEM SOLUTIONS CHAPTER 13

SOLUTION 13.1. Given

$$i''(t) + 16i'(t) + 4Bi(t) = v'(t) + 8v(t)$$

(a) with $v(t) = v'(t) = 0$ and $i_1(0) + i_2(0) = 0$

(i) the characteristic equation is

$$s^2 + 16s + 48 = 0$$

(ii) the characteristic equation has factors

(s + 4) (s + 12) = 0

and hence

$$s_1, s_2 = -4, -12$$

(iii) Equivalent circuit at t = 0

(iv) Here by KCL

$$i_1(0) = i_2(0) = 6A$$

At point (1)

$$i_1(0) + i_{2\Omega}(0) = i_2(0)$$
$$6A + i_{2\Omega}(0) = 6A$$
$$i_{2\Omega}(0) = 0$$

and

$$v_{L1} = v_{R2}(0) = 2i_{2\Omega}(0) = 0V$$

Then by KVL at t = 0

$$v_{L1}(0) - v_{6\Omega}(0) + v_{L2}(0) = 0 \qquad -v_{L2} = -36V$$

$$L_2 \frac{di_2}{dt} = -36V$$

$$i'(0) = i_2'(0) = -36A/s$$

(v) from part ii

$$i(t) = Ae^{-4t} + Be^{-12t}$$

and then

$$i'(t) = -4Ae^{-4t} - 12Be^{-12t}$$

Then using the initial conditions

$$A + B = 6$$
$$-4A - 12B = -36$$

solving yields B = 1.5 and A = 4.5. Then

$$i(t) = 4.5e^{-4t} + 1.5e^{-12t} A$$

(b) If $v(t) = 12V$ and $i(0) = i'(0) = 0$, then $v'(t) = 0$ and

$$i''(t) + 16i'(t) + 48i(t) = 8v(t)$$

with

$$i(t) = Ae^{-4t} + Be^{-12t}$$

$$i'(t) = 4Ae^{-45} - 12Be^{-12t}$$

$$i''(t) = 16Ae^{-4t} + 144Be^{-12t}$$

Then at t = ∞ when $v(\infty) = 12V$

$$i(\infty) = C,\ i'(\infty) = 0 \text{ and } i''(\infty) = 0$$

Thus

$$48C = 96 \qquad C = 2$$

Then

$$i(t) = Ae^{-4t} + Be^{-12t} + 2$$

and

$$i(0) = 0 = A + B + 2$$

$$i'(0) = 0 = -4A - 12B$$

Multiply the first of these by 4 yields

$$4A + 4B = -8$$

$$-4A - 12B = 0$$

Solving yields B = 1 and A = -3. Thus for t > 0

$$i(t) = -3e^{-4t} + e^{-12t} + 2\ A$$

(c) If v(t) is as in fig. 13.1(b) then

$$v(t) = \begin{cases} 2t & 0 < t \le 2 \\ 0 & t \ge 2 \end{cases}$$

The value of C in part (b) will change and at t = 2s, a new set of initial conditions will be required (obtainable from the solution at t = 2s) and these would be used in the decay portion described by

$$i''(t) + 16i'(t) + 48i(t) = 0 \quad t \ge 2s$$

SOLUTION 13.2.

(a) Use the figure with the currents i_1 through i_5 designated in the circuit below.

Then work from v_0 to v_{in} using repeated applications of KVL, KCL and the elemental equations:

$$i_1 = 2v_o$$
$$i_2 = 2v_o''$$
$$i_3 = i_1 + i_2 = 2(v_o' + v_o)$$

with

$$v_2 = v_o$$
$$v_1 = 0.5i_3 + v_2 = 0.5[2(v_o' + v_o) + v_o] = v_o' + 2v_o$$

then

$$i_4 = 2v_1' = 2\frac{d}{dt}(v_o^1 + 2v_o) = 2(v_o'' + 2v_o')$$
$$i_5 = i_3 + i_4 = 2(v_o' + v_o) + 2(v_o'' + 2v_o') = 2v_o'' + 6v_o' + 2v_o$$

Finally

$$v_{in} = 0.5i_5 + v_1 = 0.5(2v_o'' + 6v_o' + 2v_o) + v_o' + 2v_o = v_o'' + 4v_o' + 3v_o$$

Hence

$$\ddot{v}_{out}(t) + 4\dot{v}_{out}(t) + 3v_{out}(t) = v_{in}(t)$$

(b) Note from part (a) that

$$v_{out}(t) = v_2(t)$$

and

$$v_1(t) = \dot{v}_{out}(t) + v_{out}(t)$$

Hence

$$v_{out}(o) = v_2(o) = 1V$$
$$\dot{v}_{out}(o) = v_1(o) - v_{out}(o) = 7 - 1 = 6V$$

(c) The characteristic equation

$$s^2 + 4s + 3 = 0$$

has factors

$$(s+1)(s+3) = 0$$

and roots

$$s_1, v_2 = -1, -3$$

Thus, because of the input, $v_{in}(t) = 6V$

$$v_{out}(t) = Ae^{-t} + Be^{-3t} + C$$
$$\dot{v}_{out}(t) = -Ae^{-t} - 3Be^{-3t}$$
$$\ddot{v}_{out}(t) = Ae^{-t} + 9Be^{-3t}$$

and at t = ∞

$$\ddot{v}_{out}(\infty) + 4v_{out}(\infty) + 3v_{out}(\infty) = 6V$$
$$0 \quad + \quad 0 \quad + \quad 3C \quad = 6V$$
$$C = 2V$$

and

$$v_{out}(t) = Ae^{-t} + Be^{-3t} + 2V$$

SOLUTION 13.3. (a) From the given differential equation, the characteristic equation is

$$s^3 + 14s^2 + 52s + 24 = (s+6)(s^2 + 8s + 4) = 0$$

Therefore the roots $a = -6, b = -4 + 2\sqrt{3} = -0.5359, c = -4 - 2\sqrt{3} = -7.4641$.

(b) (i) $v(0) = v_{C3}(0) = 6$ V, as given.

(ii) To compute $v'(0)$ we write a nodal equation at node 3. In particular $(v - v_{C1}) + (v - v_{C2}) + 0.5v' + v = 0$ which implies that

$$v'(t) = 2v_{C1}(t) + 2v_{C2}(t) - 6v(t) \qquad (*)$$

Hence $v'(0) = 2v_{C1}(0) + 2v_{C2}(0) - 6v(0) = 24 + 18 - 36 = 6$ V/s.

(iii) To compute $v''(0)$ we first differentiate equation (*). This yields

$$v''(t) = 2v'_{C1}(t) + 2v'_{C2}(t) - 6\left(2v_{C1}(t) + 2v_{C2}(t) - 6v(t)\right) \qquad (**)$$

To express $v'_{C1}(t)$ and $v'_{C2}(t)$ in terms of the node voltages we write node equations at nodes 1and 2 respectively. At node 1

$$(v_{C1}(t) - v_{C2}(t)) + (v_{C1}(t) - v(t)) + 0.5v'_{C1}(t) = i(t)$$

and at node 2

$$(v_{C2}(t) - v_{C1}(t)) + (v_{C2}(t) - v(t)) + 0.5v'_{C2}(t) = 0$$

Hence

$$v'_{C1}(t) = -4v_{C1}(t) + 2v_{C2}(t)) + 2v(t) + 2i(t)$$

and

$$v_{C2}^{'}(t) = 2v_{C1}(t)) - 4v_{C2}(t) + 2v(t)$$

Substituting these two equations into (**) yields the desired result when t is set to 0. However, this quantity has no direct physical meaning.

$$v''(t) = 2\left(-4v_{C1}(t) + 2v_{C2}(t)) + 2v(t) + 2i(t)\right) + 2\left(2v_{C1}(t)) - 4v_{C2}(t) + 2v(t)\right) \\ - 6\left(2v_{C1}(t) + 2v_{C2}(t) - 6v(t)\right)$$

Hence, $v''(0)$ = 2(–48 + 18 + 12 + 2i(0)) + 2(24 – 36 + 12) – 6(24 + 18 – 36) = –72 + 4i(0) V/s^2.

Finally, using the characteristic roots found in part (a) and assuming a constant input, the form of the solution is

$$v(t) = Ae^{-6t} + Be^{-0.5359t} + Ce^{-7.4641t} + D$$

Following the methods of example 13.2,

A = 0, B = (7.3301 – 1.0774×i(0)), C = (–1.3301 + 0.0774×i(0)), and D = i(0).

(c) (*i*) Not proportional to a single voltage but it is proportional to $i_{C3}(0)$.

(*ii*) Much more complex.

(*iii*) No.

(*iv*) No. This is why we use the Laplace transform approach.

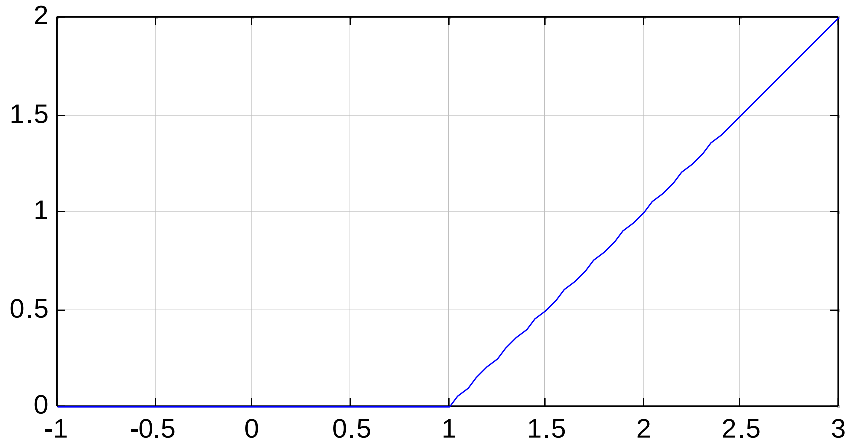
SOLUTION 13.4.

(a) $f(q + T_0)\big]_{q=t} = f(t + T_0)$

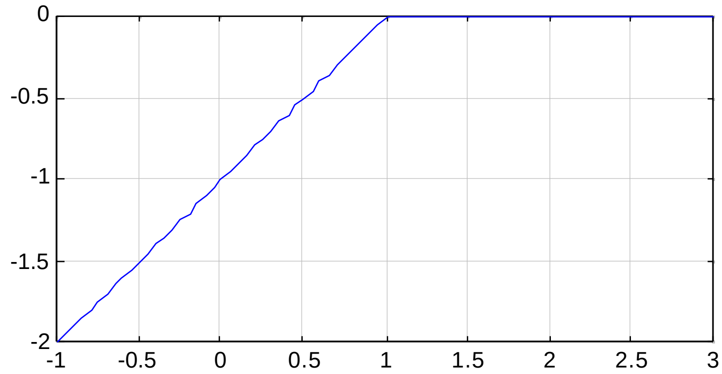
(b) $e^{-5q}\cos\left(0.5\pi q + 0.25\pi\right)\big]_{q=2t} = e^{-10t}\cos\left(\pi t + 0.25\pi\right)$

SOLUTION 13.5.

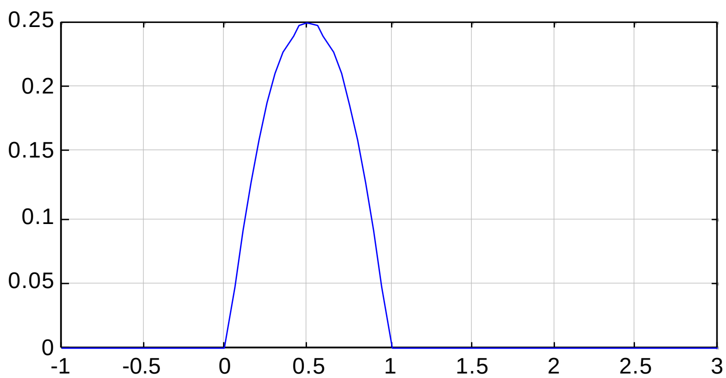
(a) Let T = 1.



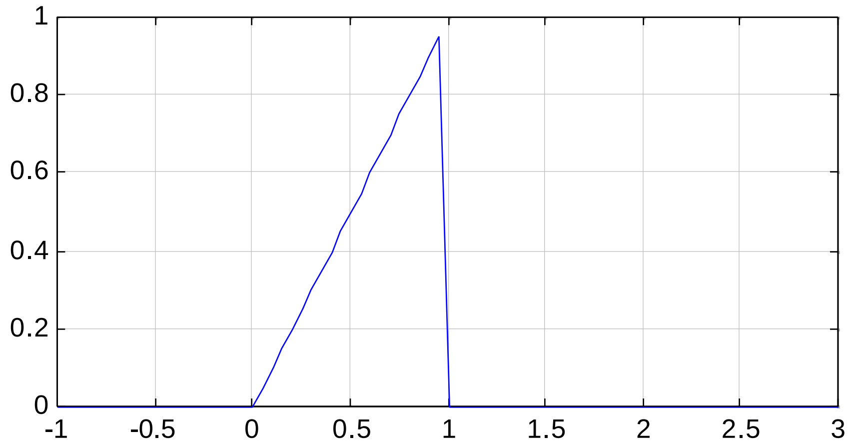
(b) Again let T = 1.



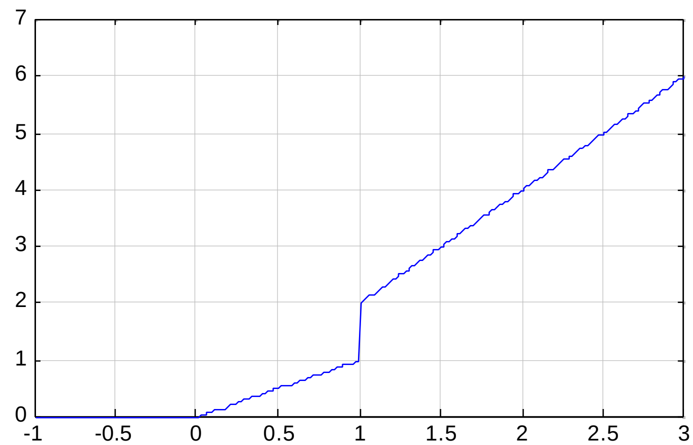
(c)



(d)

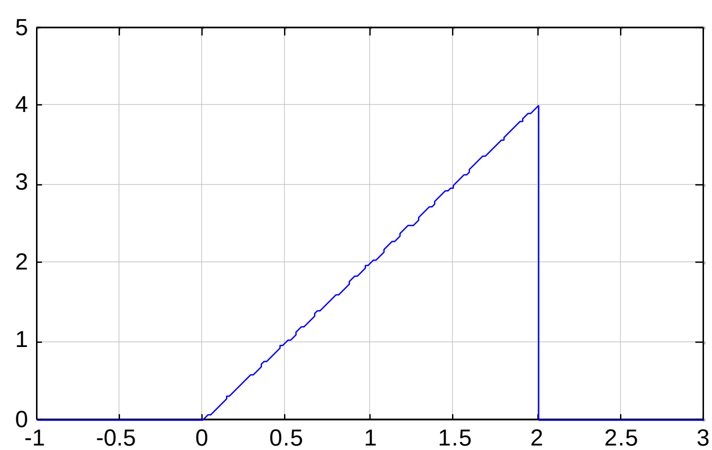


(e)



(f) $\sum_{i=0}^{r} i\delta(t-i)$

(g)



(h) Pulses of height 1 and width T.

SOLUTION 13.6.

(a) $L[f_1(t)] = F_1(s) = \int_{0^-}^{\infty} f_1(t)e^{-st}dt == \int_{T_1}^{T_2} e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{T_1}^{T_2} = \frac{1}{s}\left(e^{-sT_1} - e^{-sT_2}\right)$

(b) $f_2(t) = f_1(t)$. Hence, the answer is the same as in (a).

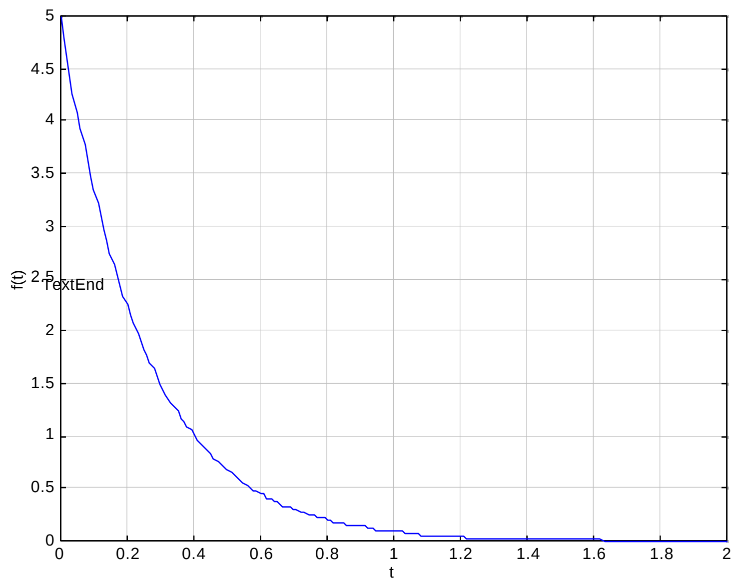
(c) $L[f_3(t)] = F_3(s) = \int_{0^-}^{\infty} f_3(t)e^{-st}dt = -2\int_{0^-}^{\infty} \delta(t)\cos(4\pi t - 0.25\pi)e^{-st}dt = -2\frac{\sqrt{2}}{2} = -\sqrt{2}$

(d) $F_4(s) = \int_{0^-}^{\infty} f_4(t)e^{-st}dt = -2\int_{0^-}^{\infty} \delta(t-T)\cos(4\pi t - 0.25\pi)e^{-st}dt = -2\cos(4\pi T - 0.25\pi)e^{-sT}$

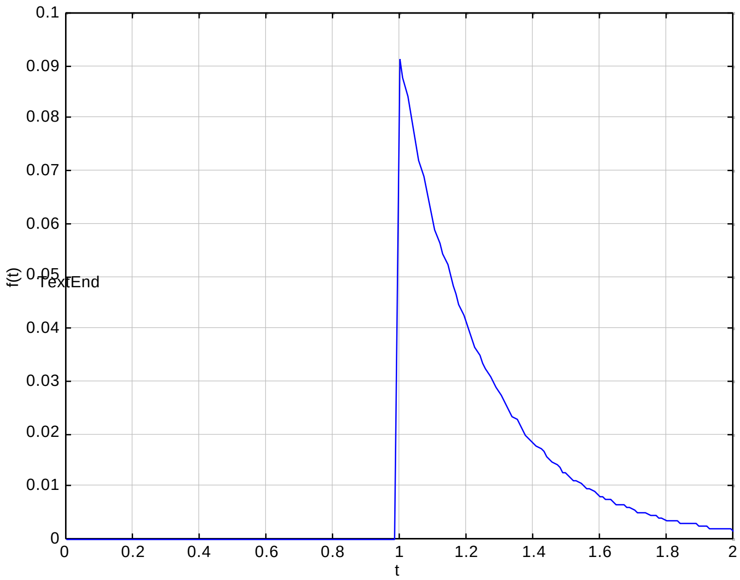
(e) $F_5(s) = \int_{0^-}^{\infty} f_5(t)e^{-st}dt = \int_{0^-}^{\infty} [\delta(t) - \delta(t-T)]e^{-st}dt = 1 - e^{-sT}$

SOLUTION 13.7.

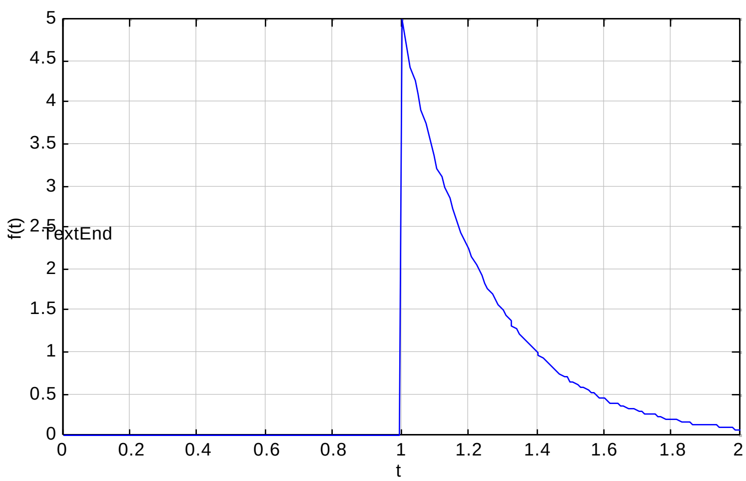
(a) $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} 5e^{-4t}e^{-st}dt = -5\left.\frac{e^{-(s+4)t}}{s+4}\right|_{0^-}^{\infty} = \frac{5}{s+4}$



(b) $F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt=\int_{1^-}^{\infty}5e^{-4t}e^{-st}dt=-5\left[\frac{e^{-(s+4)t}}{s+4}\right]_{1^-}^{\infty}=\frac{5e^{-(s+4)}}{s+4}=e^{-s}\frac{5e^{-4}}{s+4}$



(c) $F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt=\int_{1}^{\infty}5e^{-4(t-1)}e^{-st}dt=-5e^{4}\left[\frac{e^{-(s+4)t}}{s+4}\right]_{1^-}^{\infty}=e^{-s}\frac{5}{s+4}$



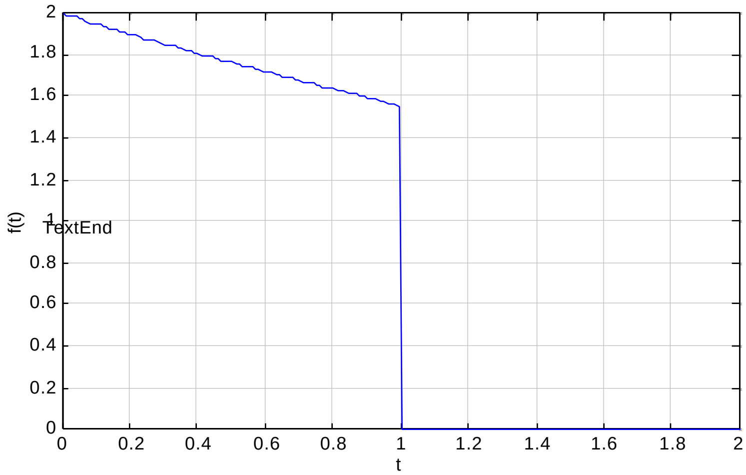
(d) $F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt=\int_{0^-}^{\infty}5e^{-4(t-1)}\delta(t)e^{-st}dt=5e^{4}$

(e) $F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt=\int_{0^-}^{\infty}5e^{-4(t-1)}\delta(t-1)e^{-st}dt=5e^{-s}$

(f)

$$F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt = 2\int_{0^-}^{\infty}\left[u(t)u(1-t)\right]e^{-0.25t}e^{-st}dt = 2\int_{0^-}^{1} e^{-0.25t}e^{-st}dt = \frac{2}{s+0.25}\left(1-e^{-(s+0.25)}\right)$$

$$= 2\int_{0^-}^{\infty} e^{-0.25t}e^{-st}dt - 2\int_{1^-}^{\infty} e^{-0.25t}e^{-st}dt = \frac{2}{s+0.25}\left(1-e^{-s}\right)$$



$$F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt =$$

SOLUTION 13.8.

(a) $F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} Ae^{\lambda t}\, e^{-st}dt = \int_{0^-}^{\infty} Ae^{-(s-\lambda)t}dt = \frac{A}{s-\lambda}$

(b) $F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} Ae^{\lambda(t-1)}\, u(t-1)e^{-st}dt = \int_{1^-}^{\infty} Ae^{-\lambda}e^{-(s-\lambda)t}dt = e^{-s}\,\frac{A}{s-\lambda}$

(c) $F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} Ae^{\lambda(t-1)}\delta(t)e^{-st}dt = Ae^{-\lambda}$

(d) $F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{1} Ae^{\lambda t}e^{-st}dt = -A\left.\frac{e^{-(s-\lambda)t}}{(s-\lambda)}\right|_{0^-}^{1} = \frac{A}{s-\lambda}\left(1-e^{-(s-\lambda)}\right)$

SOLUTION 13.9. Consider the following in which $q = at$ and $t = q/a$.

$$\mathcal{L}[f(at)] = \int_{0^-}^{\infty} f(at)e^{-st}\,dt = \frac{1}{a}\int_{0^-}^{\infty} f(q)e^{-(s/a)q}\,dq = \frac{1}{a}F\left(\frac{s}{a}\right)$$

SOLUTION 13.10.

(a)

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

Add these equations and divide by 2 to obtain

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Similarly, subtract the equations and divide by 2j to obtain

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Note that

$$L\left[e^{j\omega t}\right] = \frac{1}{s - j\omega}, \quad L\left[e^{-j\omega t}\right] = \frac{1}{s + j\omega}$$

(b)

$$L[\cos\omega t] = L\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] = \frac{0.5}{s - j\omega} + \frac{0.5}{s + j\omega} = \frac{s}{s^2 + \omega^2}$$

and

$$L[\sin\omega t] = \frac{-0.5j}{s - j\omega} + \frac{0.5j}{s + j\omega} = \frac{\omega}{s^2 + \omega^2}$$

SOLUTION 13.11. From the time frequency scaling property,

$$L\left[\sin(\omega t)\,u(t)\right] = \frac{1}{\omega}\frac{1}{\left(\frac{s}{\omega}\right)^2+1} = \frac{\omega}{s^2+\omega^2}$$

Using the time differentiation property,

$$\mathrm{L}[\omega\cos(\omega t)u(t)] = \mathrm{L}\left[\frac{d}{dt}\left(\sin(\omega t)u(t)\right)\right] = s\mathrm{L}[\sin(\omega t)\,u(t)] - \sin(0) = \frac{\omega s}{s^2+\omega^2}$$

Finally using the multiplication by t property

$$\mathrm{L}[-\omega t\cos(\omega t)u(t)] = \frac{d}{ds}\left(\frac{\omega s}{s^2+\omega^2}\right) = \frac{\omega}{s^2+\omega^2} - \frac{2\omega s^2}{\left(s^2+\omega^2\right)^2}$$

Hence

$$\mathrm{L}[\sin(\omega t)u(t) - \omega t\cos(\omega t)u(t)] = \frac{2\omega}{s^2+\omega^2} - \frac{2\omega s}{\left(s^2+\omega^2\right)^2} = \frac{2\omega\left(s^2+\omega^2\right) - 2\omega s^2}{\left(s^2+\omega^2\right)^2} = \frac{2\omega^3}{\left(s^2+\omega^2\right)^2}$$

SOLUTION 13.12. (a)

$$g_1(t) = At\sin(\omega t), \quad f(t) = \sin(t)u(t), \quad F(s) = \frac{1}{s^2+1}$$

By the frequency scaling property

$$L\left[f(\omega t)\right] = \frac{1}{\omega}\left[\frac{1}{\left(\frac{s}{\omega}\right)^2+1}\right] = \frac{1}{\omega}\left[\frac{\omega^2}{s^2+\omega^2}\right] = \frac{\omega}{s^2+\omega^2}$$

By the multiplication by t property

$$L\left[t\sin(\omega t)u(t)\right] = -\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = -\left[\frac{-2\omega s}{\left(s^2+\omega^2\right)^2}\right] = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}$$

By the linearity property

$$L\left[At\sin(\omega t)\right] = A\frac{2\omega s}{\left(s^2+\omega^2\right)^2}$$

(b)

$$g_2(t) = Ae^{at}\sin(\omega t)u(t), \quad f(t) = \sin(t)u(t), \ F(s) = \frac{1}{s^2+1}$$

By the frequency scaling property,

$$L[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

By the damping property

$$L\left[e^{at}\sin(\omega t)\right] = \frac{\omega}{(s-a)^2+\omega^2}$$

and by the linearity property

$$L[g_2(t)] = A\frac{\omega}{(s-a)^2+\omega^2}$$

(c) Before beginning, note that

$$\sin(\omega t+\theta) = \cos(\theta)\sin(\omega t) + \sin(\theta)\cos(\omega t)$$

Here

$$g_3(t) = Ae^{at}\sin(\omega t+\theta)u(t) = \cos(\theta)Ae^{at}\sin(\omega t)u(t) + \sin(\theta)Ae^{at}\cos(\omega t)u(t)$$

$$f(t) = \sin(t)u(t), \quad F(s) = \frac{1}{s^2+1}$$

From the linearity property and part (b),

$$G_3(s) = A\cos(\theta)\frac{\omega}{(s-a)^2+\omega^2} + L\{\sin(\theta)Ae^{at}\cos(\omega t)u(t)\}$$

By the differentiation in the time domain property $\cos(t) = \frac{d}{dt}\sin(t)$ which implies

$$L[\cos(t)] = s\{L[\sin(t)]\} - f(0) = \frac{s}{s^2+1} - 0 = \frac{s}{s^2+1}$$

By the frequency scaling property

$$L[\cos\omega t] = \frac{1}{\omega}\left[\frac{s/\omega}{(s/\omega)^2+1}\right] = \frac{1}{\omega^2}\left[\frac{s\omega^2}{s^2+\omega^2}\right] = \frac{s}{s^2+\omega^2}$$

Then by the linearity property

$$L\left[A\sin(\theta)\cos(\omega t)\right] = A\sin(\theta)\frac{s}{s^2+\omega^2}$$

Then by the frequency-shift property

$$L\left[A\sin(\theta)e^{at}\cos(\omega t)\right] = A\sin(\theta)\frac{s-a}{(s-a)^2+\omega^2}$$

$$G_3(s) = L\left[Ae^{at}\sin(\omega t+\theta)\right] = A\cos(\theta)\frac{\omega}{(s-a)^2+\omega^2} + A\sin(\theta)\frac{s-a}{(s-a)^2+\omega^2}$$

$$= \frac{A}{(s-a)^2+\omega^2}\left(\omega\cos(\theta)+\sin(\theta)(s-a)\right)$$

SOLUTION 13.13. (**a**) We are given that

$$f(t) = \sin(t) \qquad F(s) = \frac{1}{s^2+1}$$

And must find the transform of

$$g_1(t) = At\cos(\omega)tu(t)$$

By the time differentation property

$$\cos(t) = \frac{d}{dt}\sin(t) \qquad L\left[\cos(t)\right] = \frac{s}{s^2+1} - \sin\left(0^-\right) = \frac{s}{s^2+1}$$

By the frequency scaling property with

$$L\left[\cos(\omega t)\right] = \frac{1}{\omega}F\left(\frac{s}{\omega}\right) = \frac{1}{\omega}\left[\frac{s/\omega}{\left(s/\omega\right)^2+1}\right] = \frac{s\omega^2}{\omega^2\left(s^2+\omega^2\right)} = \frac{s}{s^2+\omega^2}$$

Using the multiplication-by-t property

$$L\left[t\cos(\omega t)\right] = -\frac{d}{ds}\left[\frac{s}{s^2+\omega^2}\right] = -\frac{s^2+\omega^2-s(2s)}{\left(s^2+\omega^2\right)^2} = \frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$$

Finally, by the linearity property

$$L\left[At\cos(\omega t)u(t)\right] = \frac{A\left(\omega^2-s^2\right)}{\left(s^2+\omega^2\right)^2}$$

(b) Let $g_2(t) = Ae^{at}\cos(\omega t)u(t)$. Recall from part (a) that

$$L\left[A\cos(\omega t)\right] = A\frac{s}{s^2+\omega^2}$$

By the Frequency-shift property

$$L\left[Ae^{at}\cos(\omega t)\right] = A\frac{s-a}{(s-a)^2+\omega^2}$$

(c) From Trig identities,

$$\cos(\omega t + \theta) = \cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta)$$

Recall from part (b) that

$$L\left[Ae^{at}\cos(\omega t)\right] = A\frac{s-a}{(s-a)^2+\omega^2}$$

Hence

$$L\left[Ae^{at}\cos(\theta)\cos(\omega t)\right] = A\cos(\theta)\frac{s-a}{(s-a)^2+\omega^2}$$

It follows that

$$L\left[Ae^{at}\cos(\omega t+\theta)\right] = A\cos(\theta)\frac{s-a}{(s-a)^2+\omega^2} - A\sin(\theta)L\left[e^{at}\sin(\omega t)\right]$$

$$= A\cos(\theta)\frac{s-a}{(s-a)^2+\omega^2} - A\sin(\theta)\frac{\omega}{(s-a)^2+\omega^2} = A\left[\frac{\cos(\theta)(s-a)-\omega\sin(\theta)}{(s-a)^2+\omega^2}\right]$$

SOLUTION 13.14. (a) We are given that

$$L\left[\cos(t)\right] = \frac{s}{s^2+1}$$

which implies by the frequency scaling property that

$$L\left[\cos(\omega t)\right] = \frac{s}{s^2+\omega^2}$$

Using the multiplication-by-t property

$$L\left[t\cos(\omega t)\right] = -\frac{d}{ds}\left[\frac{s}{s^2+\omega^2}\right] = -\frac{s^2+\omega^2-2s^2}{\left(s^2+{}^2\right)^2} = \frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$$

(b) If $g(t) = te^{at}\cos(\omega t)$, then using part (a) and frequency shift property,

$$L\left[te^{at}\cos(\omega t)\right] = \frac{(s-a)^2 - \omega^2}{\left[(s-a)^2 + \omega^2\right]^2}$$

SOLUTION 13.15. (a) With

$$\sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

$$L[\sinh(at)] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-(s-a)}{s^2-a^2}\right] = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

(b) With

$$\cosh(at) = \frac{e^{at} + e^{-at}}{2}$$

$$L[\cosh(at)] = \frac{1}{2}\left[\frac{1}{s+a} + \frac{1}{s-a}\right] = \frac{1}{2}\left[\frac{s-a+s+a}{s^2-a^2}\right] = \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

SOLUTION 13.16. (a) From Problem 15

$$L[\sinh(at)] = \frac{a}{s^2-a^2}$$

So that by the multiplication by t property

$$L[t\sinh(at)] = -\frac{d}{ds}\left[\frac{a}{s^2-a^2}\right] = -\frac{-a(2s)}{\left(s^2-a^2\right)^2} = \frac{2as}{\left(s^2-a^2\right)^2}$$

(b) From Problem 15

$$L[\cosh(at)] = \frac{s}{s^2-a^2}$$

So that by the multiplication-by-t property

$$L[t\cosh(at)] = -\frac{d}{ds}\left[\frac{s}{s^2-a^2}\right] = -\frac{\left(s^2-a^2\right)-s(2s)}{\left(s^2-a^2\right)^2} = \frac{-s^2+a^2+2s^2}{\left(s^2-a^2\right)^2} = \frac{s^2+a^2}{\left(s^2-a^2\right)^2}$$

SOLUTION 13.17. Here

$$F(s) = \frac{s+2}{s+1}$$

(a) Since $g_1(t) = 5f(t-2)$, use the time shift and linearity properties to obtain

$$L[g_1(t)] = 5e^{-2s}\left[\frac{s+2}{s+1}\right]$$

(b) Since $g_2(t) = 5e^{-2t}f(t)$, use the frequency shift and linearity properties,

$$L[g_2(t)] = 5\left[\frac{(s+2)+2}{(s+2)+1}\right] = 5\left[\frac{s+4}{s+3}\right]$$

(c) From part (a),

$$L[5f(t-2)] = 5e^{-2s}\left[\frac{s+2}{s+1}\right]$$

Therefore, since $g_3(t) = 5e^{-2t}f(t-2)$, by the frequency shift property

$$L[g_3(t)] = G_1(s+2) = 5e^{-2(s+2)}\left[\frac{(s+2)+2}{(s+2)+1}\right] = 5e^{-2(s+2)}\left[\frac{s+4}{s+3}\right]$$

(d) Since $g_4(t) = 5tf(t)$, use the multiplication-by-t and linearity principle to obtain

$$L[g_4(t)] = -\frac{d}{ds}\left[\frac{5(s+2)}{s+1}\right] = -\frac{5[(s+1)-(s+2)]}{(s+1)^2} = \frac{5}{(s+1)^2}$$

SOLUTION 13.18. In all parts

$$L[f(t)u(t)] = F(s) = \frac{s}{s^2+4}$$

(a) $$L\left[Af(t-T)u(t-T)\right] = A\frac{e^{-Ts}s}{s^2+4}$$

(b) $$L\left[Atf(t)u(t)\right] = -\frac{d}{ds}Af(s) = -A\frac{d}{ds}\left[\frac{s}{s^2+4}\right] = -A\left[\frac{\left(s^2+4\right)-s(2s)}{\left(s^2+4\right)^2}\right] = A\frac{\left(s^2-4\right)}{\left(s^2+4\right)^2}$$

(c) Note that the answer is simply a time shift of the function given in (b).

$$L\left[A(t-T)f(t-T)u(t-T)\right] = Ae^{-sT}\frac{\left(s^2-4\right)}{\left(s^2+4\right)^2}$$

(d) This function is that of part (a) multiplied by t. Hence, by the multiplication by t property,

$$L\left[Atf(t-T)u(t-T)\right] = -A\frac{d}{ds}\left[\frac{e^{-Ts}s}{s^2+4}\right] = -A\left[\frac{\left(e^{-Ts}-Tse^{-Ts}\right)(s^2+4)-2s\times se^{-Ts}}{\left(s^2+4\right)^2}\right]$$

$$= Ae^{-Ts}\left[\frac{2s^2-\left(1-Ts\right)(s^2+4)}{\left(s^2+4\right)^2}\right] = Ae^{-Ts}\left[\frac{Ts^3+s^2+4Ts-4}{\left(s^2+4\right)^2}\right]$$

SOLUTION 13.19. In all parts

$$F(s) = L\left[f(t)u(t)\right] = \frac{\omega}{s^2+\omega^2}$$

(a) By the time shift property, $L\left[Af(t-T)u(t-T)\right] = A\frac{\omega e^{-Ts}}{\left(s^2+\omega^2\right)}$

Using the multiplication-by-t property,

$$L\left[Atf(t)u(t)\right] = -A\frac{d}{ds}\left[F(s)\right] = -A\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = A\frac{\omega(2s)}{\left(s^2+\omega^2\right)^2} = A\frac{2\omega s}{\left(s^2+\omega^2\right)^2}$$

© The answer here is an application of the time shift property to the answer of part (b).

$$L\left[A(t-T)f(t-T)u(t-T)\right] = Ae^{-sT}\frac{2\omega s}{\left(s^2+\omega^2\right)^2}$$

(d) The answer here uses the multiplication-by-t property applied to the answer of part (a).

$$L\left[Atf(t-T)u(t-T)\right] = -A\omega\frac{d}{ds}\left[\frac{e^{-Ts}}{\left(s^2+\omega^2\right)}\right] = A\omega\left[\frac{Te^{-Ts}(s^2+\omega^2)+2se^{-Ts}}{\left(s^2+\omega^2\right)^2}\right]$$

$$= A\omega e^{-Ts}\left[\frac{Ts^2+2s+T\omega^2}{\left(s^2+\omega^2\right)^2}\right]$$

Solution 13.20. (a)

$$f(2t) = \delta(2t)+\delta(2t-1)$$

For the first term on the right, the peak occurs at t = 0 and

$$\int_{0^-}^{0^+}\delta(2t)dt = 0.5\int_{0^-}^{0^+}\delta(\tau)d\tau = 0.5$$

under the transformation $\tau = 2t$. For the second term, the function peaks at t = 0.5 and

$$\int_{0.5^-}^{0.5^+}\delta(2t-1)dt = 0.5\int_{0^-}^{0^+}\delta(\tau)d\tau = 0.5$$

under the transformation $\tau = 2t - 1$. Hence

$$f(2t) = \delta(2t)+\delta(2t-1) = 0.5\left[\delta(t)+\delta(t-0.5)\right]$$

Therefore, a = 0.5 = b.

(b) (i) Here $F(s) = L[f(t)] = 1 + e^{-s}$. By the time scaling property

$$L[f(2t)] = \frac{1}{2}F(s/2) = 0.5\left(1 + e^{-0.5s}\right)$$

(b)-(ii) For this part,

$$L[f(2t)] = 0.5L[\delta(t) + \delta(t - 0.5)] = 0.5\left(1 + e^{-0.5s}\right)$$

SOLUTION 13.21. (a)

$$L[v(t)] = 2L[g''(t)] - L[g'(t)] = 2s^2F(s) - 2sg(0^-) - 2g'(0^-) - sF(s) + g(0^-)$$

$$= \left(2s^2 - s\right)F(s) - 2sg(0^-) - 2g'(0^-) + g(0^-) = \frac{(2s-1)(s+1)}{s^2}$$

(b)

$$L[v(t)] = 2L[f''(t)] - L[f'(t)] = 2s^2F(s) - sF(s) - 2sf(0^-) - 2f'(0^-) + f(0^-)$$

$$= \left(2s^2 - s\right)F(s) - 2s - 2\lambda + 1 = \frac{(2s-1)(s+1)}{s^2} - 2s - 2\lambda + 1$$

(c)

$$L[v(t)] = L[g'(t)] - L\left[\int_{-\infty}^{t} g(q)dq\right] = sG(s) - g(0^-) - \frac{G(s)}{s} - \frac{1}{s}\int_{-\infty}^{0^-} g(q)dq = \left(s - \frac{1}{s}\right)G(s)$$

$$= \left(s - \frac{1}{s}\right)\frac{s+1}{s^3} = (s+1)\left[\frac{1}{s^2} - \frac{1}{s^4}\right]$$

(d)

$$L[v(t)] = L[f'(t)] - L\left[\int_{-\infty}^{t} f(q)dq\right] = sF(s) - f(0^-) - \frac{F(s)}{s} - \frac{1}{s}\int_{-\infty}^{0^-} f(q)dq$$

$$= (s+1)\left[\frac{1}{s^2} - \frac{1}{s^4}\right] - 1 - \frac{\lambda^{-1}}{s}$$

assuming $\lambda > 0$. The expression is ill-defined if $\lambda \le 0$.

SOLUTION 13.22.

(a) (i) If

$$F(s) = L[f(t)u(t)] = \ell n\left(\frac{s^2+4}{s^2}\right) = \ell n\left(s^2+4\right) - \ell n\left(s^2\right)$$

Then by the multiplication-by-t property

$$L[-tf(t)u(t)] = +\frac{d}{ds}\left[\ell n\left(s^2+4\right) - \ell n\left(s^2\right)\right]$$

$$= \frac{2s}{s^2+4} - \frac{2s}{s^2} = \frac{2s}{s^2+4} - \frac{2}{s} = \frac{2s^2-2s^2-8}{s\left(s^2+4\right)} = \frac{-8}{s\left(s^2+4\right)}$$

(ii) Using the solution to (a)-(i), by the frequency shift property

$$L\left[-te^{-2t}f(t)u(t)\right] = \frac{-8}{(s+2)\left((s+2)^2+4\right)} = \frac{-8}{(s+2)\left(s^2+4s+8\right)}$$

$$= \frac{-8}{s^3+6s+16s+16}$$

(b) If

$$G(s) = \frac{-8}{s\left(s^2+4\right)}$$

a partial fraction expansion may be employed

$$G(s) = \frac{-8}{s\left(s^2+4\right)} = \frac{K_1}{s} + \frac{As+B}{s^2+4} = \frac{-2}{s} + \frac{2}{s^2+4}$$

Hence,

$$g(t) = [2\cos(2t) - 2]u(t)$$

and

$$f(t) = -\frac{g(t)}{t} = \frac{2}{t} - \frac{2}{t}\cos(2t)$$

SOLUTION 13.23: Part (a)-(i): From table 13.2, the multiplication by t property implies that

$$L[-tf(t)u(t)] = \frac{d}{ds}F(s) = \frac{d}{ds}\ell n\left[\frac{s+a}{s-a}\right] = \frac{d}{ds}\left(\ell n[s+a] - \ell n[s-a]\right) = \frac{1}{s+a} - \frac{1}{s-a} = \frac{-2a}{s^2-a^2}$$

Part (a)-(ii): Let us make use of the answer to part (a)-1. Let $G(s) = L[-tf(t)u(t)] = \frac{-2a}{s^2-a^2}$.

Then by the frequency shift property in table 13.2,

$$L[-te^{-at}f(t)u(t)] = L[e^{-at}\left(-tf(t)u(t)\right)] = G(s+a) = \frac{-2a}{(s+a)^2-a^2} = \frac{-2a}{s(s+2a)}$$

Part (b): $g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{-2a}{s^2-a^2}\right] = L^{-1}\left[\frac{1}{s+a}\right] - L^{-1}\left[\frac{1}{s-a}\right] = \left(e^{-at} - e^{at}\right)u(t)$

More specifically,

$$g(t) = \left(e^{-at} - e^{at}\right)u(t) = -2\left(\frac{e^{at} - e^{-at}}{2}\right)u(t) = -2\sinh(at)\,u(t)$$

Hence $f(t) = \frac{g(t)}{-t} = \frac{2\sinh(at)}{t}$.

SOLUTION 13.24.

(a)-(i) If

$$F(s) = L\left[f(t)u(t)\right] = \ell n\left(\frac{s+a}{s+b}\right) = \ell n(s+a) - \ell n(s+b)$$

Then by the multiplication-by-t property

$$L\left[-tf(t)u(t)\right] = +\frac{d}{ds}\left[\ell n(s+a) - \ell n(s+b)\right] = \frac{1}{s+a} - \frac{1}{s+b} = \frac{s+b-(s+a)}{(s+a)(s+b)} = \frac{b-a}{(s+a)(s+b)}$$

(ii) By the frequency-shift property

$$L\left[-te^{-at}f(t)u(t)\right] = \frac{b-a}{(s+2a)(s+a+b)} = \frac{b-a}{s^2+s(3a+b)+2a(a+b)}$$

(b) If

$$G(s) = L[-tf(t)u(t)] = \frac{b-a}{(s+a)(s+b)}$$

a partial fraction expansion may be employed

$$G(s) = \frac{b-a}{(s+a)(s+b)} = \frac{K_1}{s+a} + \frac{K_2}{s+b} = \frac{1}{s+a} + \frac{-1}{s+b}$$

Hence, $g(t) = \left(e^{-at} - e^{-bt}\right)u(t)$

and

$$f(t) = \frac{g(t)}{-t} = \frac{e^{-bt}}{t} - \frac{e^{-at}}{t}$$

SOLUTION 13.25. The relationship is $f(t) = \frac{d}{dt}g(t)$ or equivalently, $g(t) = \int_{-\infty}^{t} f(q)\,dq$.

Now we have that $f(t) = 6\delta(t) - 12\delta(t-2) + 6\delta(t-4)$. Therefore,

$$F(s) = 6 - 12e^{-2s} + 6e^{-4s}$$

From the time integration property,

$$G(s) = \frac{F(s)}{s} = \frac{6}{s} - \frac{12e^{-2s}}{s} + \frac{6e^{-4s}}{s}$$

SOLUTION 13.26. For 0 ≤ t < T_1, we see that $g(t) = \int_{-\infty}^{t} f(q)\,dq$. Thus one presupposes here that the relationship is $f(t) = \frac{d}{dt}g(t)$ or equivalently, $g(t) = \int_{-\infty}^{t} f(q)\,dq$. As such E = A – B and D = A – B + C.

Further, $f(t) = A\delta(t) - B\delta(t-T_1) + C\delta(t-T_2)$ which implies that

$$F(s) = A - Be^{-T_1 s} + Ce^{-T_2 s}$$

Thus

$$G(s) = \frac{F(s)}{s} = \frac{A}{s} - \frac{Be^{-T_1 s}}{s} + \frac{Ce^{-T_2 s}}{s}$$

SOLUTION 13.27.

(a) $f(t) = u(t) + u(t-1)$ $\qquad f(s) = \frac{1}{s} + \frac{e^{-s}}{s} = \frac{1}{s}\left(1 - e^{-s}\right)$

(b)

$f(t) = u(t) + u(t-1) - u(t-3)$ $\qquad f(s) = \frac{1}{s} + \frac{e^{-s}}{s} - \frac{e^{-3s}}{s} = \frac{1}{s}\left(1 + e^{-s} - e^{-3s}\right)$

(c)

$f(t) = u(t) + u(t-1) - 2u(t-3)$ $\qquad F(s) = \frac{1}{s} + \frac{e^{-s}}{s} - \frac{2e^{-3s}}{s} = \frac{1}{s}\left(1 + e^{-s} - 2e^{-3s}\right)$

(d)

$f(t) = 2u(t) - u(t-2) - u(t-3)$ $\qquad F(s) = \frac{2}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} = \frac{1}{s}\left(2 - e^{-2s} - e^{-3s}\right)$

SOLUTION 13.28.

(a) $f(t) = 2r(t) - 2r(t-1)$ $\qquad F(s) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2} = \frac{2}{s^2}\left(1 - e^{-s}\right)$

(b) $f(t) = 2r(t) - 2r(t-1) + r(t-2)$ $\qquad F(t) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} = \frac{1}{s^2}\left(2 - 2e^{-s} + e^{-2s}\right)$

(c) $f(t) = 2r(t) - 2r(t-1) - 2r(t-2) + 2r(t-3)$. It follows that

$$F(s) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{2e^{-3s}}{s^2} = \frac{2}{s^2}\left(1 - e^{-s} - e^{-2s} + e^{-3s}\right)$$

SOLUTION 13.29. (a) Here $f(t) = \frac{3}{2}r(t) - 3r(t-2) + \frac{3}{2}r(t-4)$. Thus,

$$F(s) = \frac{3}{2s^2} - \frac{3e^{-2s}}{s^2} + \frac{3e^{-4s}}{2s^2} = \frac{3}{2s^2}\left(1 - 2e^{-2s} + e^{-4s}\right)$$

(b) Here $f(t) = \frac{V_o}{T}r(t) - \frac{2V_o}{T}r(t-T) + \frac{V_o}{T}r(t-2T)$. Thus

$$F(s) = \frac{V_o}{T}\left(\frac{1}{s^2} - \frac{2e^{-T}}{s^2} + \frac{e^{-2T}}{s^2}\right)$$

(c) Here $f(t) = 2r(t-1) - 4r(t-2) + 4r(t-4) - 2r(t-5)$

$$F(s) = \frac{1}{s^2}\left(2e^{-s} - 4e^{-2s} + 4e^{-4s} - 2e^{-5s}\right)$$

SOLUTION 13.30.

(a) Here $f(t) = 2r(t) - 2r(t-1) - 2u(t-4)$ implies

$$F(s) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{2e^{-4s}}{s} = \frac{2}{s^2}\left(1 - e^{-s} - se^{-4s}\right)$$

(b) Here $f(t) = 2u(t) - r(t-2) + r(t-4)$ implies

$$F(s) = \frac{2}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-4s}}{s^2} = \frac{2}{s^2}\left(s - e^{-2s} - e^{-4s}\right)$$

SOLUTION 13.31. **(a)** Here $f(t) = 2u(t) - r(t) + 2r(t-2) - 2r(t-4) - 2u(t-4)$

Thus $F(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{2e^{-2s}}{s^2} - \frac{2e^{-4s}}{s^2} - \frac{2e^{-4s}}{s}$.

(b) $f(t) = -u(t) + r(t) - r(t-2) - u(t-2)$. Hence $F(s) = \frac{-1}{s} + \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s}$.

(c) $f(t) = 2r(t) - 2r(t-1) - 2u(t-1) + 2u(t-2) - 2r(t-2) + 2r(t-3)$.

Hence $F(s) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{2e^{-s}}{s} + \frac{2e^{-2s}}{s} - \frac{2e^{-2s}}{s^2} + \frac{2e^{-3s}}{s^2}$.