

PROBLEM SOLUTIONS

Solution 16.1.

(a) By the definition of the convolution integral

$$f_2(t) \quad f_2(t) = \int_{-\infty}^{\infty} f_2(t-\tau) f_2(\tau) d\tau = \int_{-\infty}^{\infty} 2u(t-\tau) 2u(\tau) d\tau = 4\int_{0}^{\infty} u(t-\tau) d\tau$$

The integrand, $u(t-\tau)$, is nonzero only when $\tau \le t$. This suggests that there are two regions of consideration: $t < 0$ and $t \ge 0$.

Case 1: $t < 0$. Here $u(t-\tau) = 0$ since τ is restricted to the interval $[0,\infty)$. Hence

$$f_2(t) \quad f_2(t) = 0, \text{ for } t < 0.$$

Case 2: $t \ge 0$.

$$f_2(t) \quad f_2(t) = 4\int_{-\infty}^{\infty} u(t-\tau) d\tau = 4\int_{0}^{t} d\tau = 4t, \text{ for } t \ge 0.$$

In sum,

$$f_2(t) \quad f_2(t) = \begin{cases} 0, & t < 0 \\ 4t, & t \ge 0 \end{cases}$$

(b) By the definition of the convolution integral

$$f_2(t) \quad f_3(t) = \int_{-\infty}^{\infty} f_2(t-\tau) f_3(\tau) d\tau = \int_{-\infty}^{\infty} 2u(t-\tau) 4e^{-2\tau} u(\tau) d\tau = 8\int_{0}^{\infty} e^{-2\tau} u(t-\tau) d\tau$$

The integrand, $u(t-\tau)$, is nonzero only when $\tau \le t$. This suggests that there are two cases to consider: $t < 0$ and $t \ge 0$.

Case 1: $t < 0$. Here $u(t-\tau) = 0$ since τ is restricted to the interval $[0,\infty)$. Hence

$$f_2(t) \quad f_3(t) = 0, \text{ for } t < 0.$$

Case 2: $t \ge 0$.

$$f_2(t) \quad f_3(t) = 8\int_{0}^{t} e^{-2\tau} d\tau = -4e^{-2\tau}\Big]_0^t = 4(1 - e^{-2t})$$

In sum,

$$f_2(t) \quad f_2(t) = \begin{cases} 0, & t < 0 \\ 4(1 - e^{-2t}), & t \ge 0 \end{cases}$$

(c) By the definition of the convolution integral and the sifting property of the delta function

$$f_1(t) \quad f_3(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_3(\tau) d\tau = \int_{-\infty}^{\infty} 5\delta(t-\tau) 4e^{-2\tau} u(\tau) d\tau =$$

$$= 20e^{-2\tau} u(\tau)\Big]_{\tau = t} = 20e^{-2t} u(t)$$

(d) By the definition of the convolution integral

$$f_3(t) \star f_3(t) = \int_{-\infty}^{\infty} f_3(t-\tau) f_3(\tau) d\tau = \int_{-\infty}^{\infty} 4e^{-2(t-\tau)} u(t-\tau) 4e^{-2\tau} u(\tau) d\tau = 16e^{-2t} \int_{0}^{\infty} u(t-\tau) d\tau$$

The integrand, $u(t-\tau)$, is nonzero only when $\tau \leq t$. This suggests that there are two cases to consider: $t < 0$ and $t \geq 0$.
Case 1: $t < 0$. Here $u(t-\tau) = 0$ since τ is restricted to the interval $[0, \infty)$. Hence

$$f_3(t) \star f_3(t) = 0, \text{ for } t < 0.$$

Case 2: $t \geq 0$.

$$f_3(t) \star f_3(t) = 16e^{-2t} \int_{0}^{t} d\tau = 16te^{-2t}$$

In sum,

$$f_3(t) \star f_3(t) = \begin{cases} 0, & t < 0 \\ 16te^{-2t}, & t \geq 0 \end{cases}$$

(e) By the definition of the convolution integral and the sifting property of the delta function

$$f_1(t+2) \star f_2(t+4) = \int_{-\infty}^{\infty} f_1(t+2-\tau) f_2(\tau+4) d\tau = \int_{-\infty}^{\infty} 5\delta(t+2-\tau) 2u(\tau+4) d\tau =$$
$$= 10u(\tau+4)\big]_{\tau=t+2} = 10u(t+6)$$

(f) By the distributive property of convolution

$$f_2(t) \star \left[f_2(t) + f_3(t) \right] = f_2(t) \star f_2(t) + f_2(t) \star f_3(t)$$

Using the results of parts (a) and (b) the result follows immediately

$$f_2(t) \star \left[f_2(t) + f_3(t) \right] = \begin{cases} 0, & t < 0 \\ 4(1 + t - e^{-2t}), & t \geq 0 \end{cases}$$

Solution 16.2.
(a) By definition

$$f_3(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau = \int_{-\infty}^{\infty} K_1 u(\tau - T_1) K_2 u(t - \tau - T_2) d\tau = K_1 K_2 \int_{T_1}^{\infty} u(t - \tau - T_2) d\tau$$

Here observe that $u(t-\tau-T_2) = 0$ for $\tau > t - T_2$. Hence there are two cases to consider: $t - T_2 < T_1$ and $t - T_2 \geq T_1$.

Case 1: $t - T_2 < T_1$. Here $u(t - \tau - T_2) = 0$, since τ is restricted to the interval $[T_1, \infty)$.

$$f_3(t) = 0$$

Case 2: $t - T_2 \geq T_1$.

$$f_3(t) = K_1K_2 \int_{T_1}^{t-T_2} d\tau = K_1K_2(t - T_2 - T_1)$$

In sum,

$$f_3(t) = \begin{cases} 0, & t < T_1 + T_2 \\ K_1K_2(t - T_2 - T_1), & t \ge T_1 + T_2 \end{cases}$$

(b) By definition

$$f_3(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau = \int_{-\infty}^{\infty} K_1 u(t - \tau + T_1) K_2 u(\tau + T_2) d\tau = K_1K_2 \int_{-T_2}^{\infty} u(t - \tau + T_1) d\tau$$

Here observe that $u(t - \tau + T_1) = 0$ for $\tau > t + T_1$. Hence there are two cases to consider: $t + T_1 < -T_2$ and $t + T_1 \ge -T_2$.

Case 1: $t + T_1 < -T_2$. Here $u(t - \tau + T_1) = 0$, since τ is restricted to the interval $[-T_2, \infty)$: $f_3(t) = 0$.

Case 2: $t + T_1 \ge -T_2$.

$$f_3(t) = K_1K_2 \int_{-T_2}^{t+T_1} d\tau = K_1K_2(t + T_1 + T_2)$$

In sum,

$$f_3(t) = \begin{cases} 0, & t < -T_1 - T_2 \\ K_1K_2(t - T_2 - T_1), & t \ge -T_1 - T_2 \end{cases}$$

(c) By definition

$$f_3(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau = \int_{-\infty}^{\infty} K_1 u(t - \tau) K_2 e^{-a\tau} u(\tau) d\tau = K_1K_2 \int_0^{\infty} e^{-a\tau} u(t-\tau) d\tau$$

The integrand is nonzero only when $\tau \le t$. Hence, there are two cases to consider: $t < 0$ and $t \ge 0$.

Case 1: $t < 0$. $f_3(t) = 0$, for $t < 0$.
Case 2: $t \ge 0$.

$$f_3(t) = K_1K_2 \int_0^t e^{-a\tau} d\tau = K_1K_2 \left[\frac{e^{-a\tau}}{-a} \right]_0^t = \frac{K_1K_2}{a}(1 - e^{-at}).$$

Therefore

$$f_3(t) = \frac{K_1K_2}{a}(1 - e^{-at}) u(t). \text{ for } t \ge 0.$$

(d) By definition

$$f_3(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau = \int_{-\infty}^{\infty} K_1 u(t - \tau + T_1) K_2 e^{-a\tau} u(\tau) d\tau = K_1K_2 \int_0^{\infty} e^{-a\tau} u(t - \tau + T_1) d\tau$$

The integrand is nonzero only when $\tau \le t + T_1$. Hence, there are two cases to consider: $t + T_1 < 0$ and $t + T_1 \ge 0$.

Case 1: $t + T_1 < 0$. Here $u(t - \tau + T_1) = 0$, since τ is restricted to the interval $[0,\infty)$. Hence $f_3(t) = 0$.

Case 2: $t + T_1 \ge 0$.

$$f_3(t) = K_1K_2 \int_0^{t+T_1} e^{-a\tau} d\tau = K_1K_2 \left[\frac{e^{-a\tau}}{-a}\right]_0^{t+T_1} = \frac{K_1K_2}{a}\left[1 - e^{-a(t+T_1)}\right].$$

Therefore

$$f_3(t) = \frac{K_1K_2}{a}\left[1 - e^{-a(t+T_1)}\right]u(t+T_1).$$

(e) By definition

$$f_3(t) = \int_{-\infty}^{\infty} f_1(t-\tau)\, f_2(\tau)\, d\tau = \int_{-\infty}^{\infty} K_1 u(-t+\tau) K_2 e^{-a\tau} u(\tau) d\tau = K_1K_2 \int_0^{\infty} e^{-a\tau} u(-t+\tau)\, d\tau$$

The integrand is nonzero only when $\tau \ge t$. Hence, there are two cases to consider: $t \le 0$ and $t > 0$.
Case 1: $t \le 0$. Here $u(-t+\tau) = 1$, since $\tau \ge 0$. Hence

$$f_3(t) = K_1K_2 \int_0^{\infty} e^{-a\tau} d\tau = K_1K_2 \left[\frac{e^{-a\tau}}{-a}\right]_0^{\infty} = \frac{K_1K_2}{a}, \text{ for } t \le 0.$$

Case 2: $t > 0$.

$$f_3(t) = K_1K_2 \int_t^{\infty} e^{-a\tau} d\tau = K_1K_2 \left[\frac{e^{-a\tau}}{-a}\right]_t^{\infty} = \frac{K_1K_2}{a} e^{-at}, \text{ for } t > 0.$$

In sum,

$$f_3(t) = \begin{cases} \dfrac{K_1K_2}{a}, & t \le 0 \\ \dfrac{K_1K_2}{a} e^{-at}, & t > 0 \end{cases}$$

Solution 16.3.
(a) By definition

$$f_3(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau = \int_{-\infty}^{\infty} K_1 e^{-a\tau} u(\tau) K_2 e^{-a(t-\tau)} u(t-\tau) d\tau = K_1K_2 \int_0^{\infty} e^{-at} u(t-\tau)\, d\tau$$

The integrand, $u(t-\tau)$, is nonzero only when $\tau \ge t$. Hence, there are two cases to consider: $t < 0$ and $t \ge 0$.

Case 1: $t < 0$. Here $u(t-\tau) = 0$, since $\tau \ge 0$. Hence $f_3(t) = 0$, for $t < 0$.
Case 2: $t \ge 0$.

$$f_3(t) = K_1 K_2 e^{-at} \int_0^t d\tau = K_1 K_2 e^{-at} t, \text{ for } t \geq 0.$$

In sum,

$$I_c(s) = \frac{Cs}{Cs + \frac{1}{R}} I_{in}(s)$$

(b) By definition

$$a = 1$$
$$K = 1$$

The integrand, $u(t-\tau)$, is nonzero only when $\tau \geq t$. Hence, there are two cases to consider: $t < 0$ and $t \geq 0$.

Case 1: $t < 0$. Here $u(t-\tau) = 0$, since $\tau \geq 0$. Hence $f_3(t) = 0$, for $t < 0$.
Case 2: $t \geq 0$.

$$f_3(t) = K_1 K_2 e^{-bt} \int_0^t e^{(b-a)\tau} d\tau = \begin{cases} t & \text{if } a = b \\ \dfrac{1}{b-a}\left[e^{(b-a)t} - 1\right] & \text{if } a \neq b \end{cases}$$

Therefore, for $t \geq 0$,

$$f_3(t) = \begin{cases} K_1 K_2 e^{-bt} t u(t) & \text{if } a = b \\ \dfrac{K_1 K_2}{b-a}\left[e^{-at} - e^{-bt}\right] & \text{if } a \neq b \end{cases}$$

(c) By replacing $K_1 = 50$, $K_2 = 20$ and $a = 10$ in the formula of $f_3(t)$ in part (a) the answer for part (i) is easily obtained as

$$f_3(t) = \begin{cases} 0, & t < 0 \\ 1000 e^{-10t} t, & t \geq 0 \end{cases}$$

For part (ii) the parameters have the following values: $v_c(t) = \int_{-\infty}^{t} e^{-(t-\tau)} d\tau = e^{\tau - t}\Big]_{-\infty}^{t} = 1$, $K_2 = 0.2$, $a = 10$ and $b = 0.2$. Using these values in the formula developed in part (b) for $f_3(t)$ the answer follows immediately

$$f_3(t) = 0.102\left(e^{-0.2t} - e^{-10t}\right) u(t)$$

SOLUTION 16.4.

(a) Using the impulse response theorem and the definition of the convolution integral the response of the system, $y(t)$, can be computed as follows

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau) v(\tau) d\tau = \int_{-\infty}^{\infty} 2e^{-2(t-\tau)} u(t-\tau)\left[u(\tau-1) - u(\tau-3)\right] d\tau$$

Observing that $u(\tau-1) - u(\tau-3)$ is nonzero only when $1 \leq \tau < 3$ yields

$$y(t) = 2\int_1^3 e^{2(\tau - t)} u(t-\tau) d\tau$$

The integrand in the above equation is nonzero only when $\tau \le t$. This suggests three regions of consideration: $t < 1$, $1 \le t \le 3$, and $3 < t$.

Case 1: $t < 1$. Here $u(t-\tau) = 0$, since τ is restricted to the interval [1,3]. Hence $y(t) = 0$, for $t < 1$.

Case 2: $1 \le t \le 3$.

$$y(t) = 2\int_1^t e^{2(\tau - t)} d\tau = e^{2(\tau - t)}\Big]_1^t = 1 - e^{2(1-t)}, \text{ for } 1 \le t \le 3$$

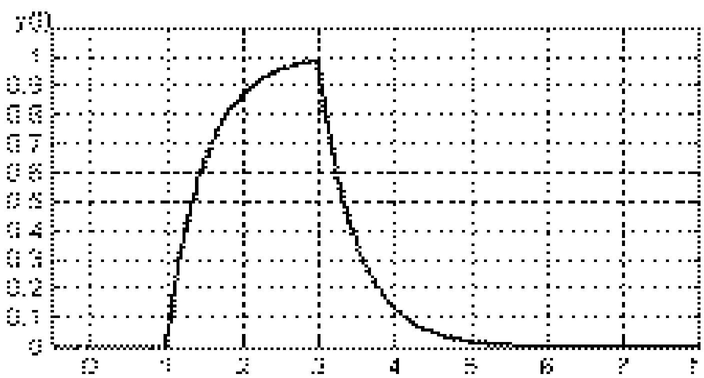
Case 3: $3 < t$.

$$y(t) = 2\int_1^3 e^{2(\tau - t)} d\tau = e^{2(\tau - t)}\Big]_1^3 = e^{2(1-t)}(e^4 - 1), \text{ for } 3 < t$$

In sum,

$$y(t) = \begin{cases} 0, & t < 1 \\ 1 - e^{2(1-t)}, & 1 \le t \le 3 \\ e^{2(1-t)}(e^4 - 1), & 3 < t \end{cases}$$

A picture of $y(t)$ is sketched in the next figure.



(b) Using the impulse response theorem and the definition of the convolution integral the response of the system, $y(t)$, can be computed as follows

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)v(\tau)d\tau = \int_{-\infty}^{\infty} 2e^{-2(t-\tau)} u(t-\tau)u(2-t+\tau)\left[u(\tau-1) - u(\tau-3)\right]d\tau$$

Observing that $u(\tau - 1) - u(\tau - 3)$ is nonzero only when $1 \le \tau < 3$ yields

$$y(t) = 2\int_1^3 e^{2(\tau - t)} u(t-\tau)u(2-t+\tau) d\tau$$

The integrand in the above equation is nonzero only when $t - 2 \le \tau \le t$. This suggests four regions of consideration: $t < 1$, $1 \le t \le 3$, $3 < t \le 5$ and $5 < t$.

Case 1: $t < 1$. Here $u(t-\tau) = 0$, since τ is restricted to the interval [1,3]. Hence $y(t) = 0$, for $t < 1$.

Case 2: $1 \le t \le 3$. Here $u(t-\tau)u(2-t+\tau)$ is nonzero only when $1 \le \tau \le t$. Therefore,

$$y(t) = 2\int_{1}^{t} e^{2(\tau - t)} d\tau = e^{2(\tau - t)}\Big]_{1}^{t} = 1 - e^{2(1-t)}, \text{ for } 1 \le t \le 3.$$

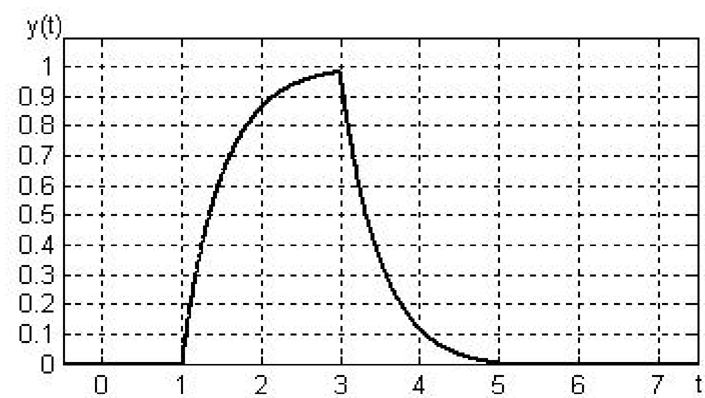
Case 3: $3 < t \le 5$. Here $u(t-\tau)u(2-t+\tau)$ is nonzero only when $t-2 \le \tau \le 3$. Therefore,

$$y(t) = 2\int_{t-2}^{3} e^{2(\tau - t)} d\tau = e^{2(\tau - t)}\Big]_{t-2}^{3} = e^{2(3-t)} - e^{-4}, \text{ for } 3 < t \le 5.$$

Case 4: $5 < t$. Here $u(t-\tau)u(2-t+\tau) = 0$, since τ is restricted to the interval [1,3]. Therefore,

$$y(t) = 0, \text{ for } 5 < t.$$

A picture of $y(t)$ is sketched in the next figure.



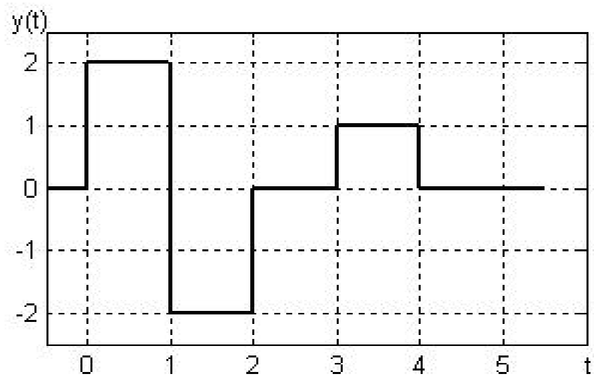
(c) By the impulse response theorem, the zero-state response of the circuit $y(t)$ is

$$y(t) = h(t)\ \ v(t)$$

Using the definition of the convolution integral and the sifting property of delta function it follows that

$$y(t) = 2h(t) - 2h(t-1) + h(t-2) = \begin{cases} 0, & t < 0 \\ 2, & 0 \le t < 1 \\ -2, & 1 \le t < 2 \\ 0, & 2 \le t < 3 \\ 1, & 3 \le t < 4 \\ 0, & 4 \le t \end{cases}$$

Using the waveform of $h(t)$ given in figure P16.4, $y(t)$ is sketched in the next picture.



SOLUTION 16.5.

(a) By definition

$$f_4(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau = \int_{-\infty}^{\infty} \delta(t-\tau-2)\ 2\ u(\tau+1) d\tau$$

By the sifting property of delta function it follows that $f_4(t) = 2u(\tau+1)\big]_{\tau=t-2} = 2u(t-1)$.

(b) By the definition of convolution and the sifting property of delta function

$$f_5(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_3(\tau) d\tau = \int_{-\infty}^{\infty} \delta(t-\tau-2) e^{-2\tau} u(\tau) d\tau$$
$$= e^{-2\tau} u(\tau)\big]_{\tau=t-2} = e^{-2(t-2)} u(t-2)$$

(c) By definition

$$f_6(t) = \int_{-\infty}^{\infty} f_2(t-\tau) f_3(\tau) d\tau = \int_{-\infty}^{\infty} 2u(t-\tau+1) e^{-2\tau} u(\tau) d\tau = 2\int_{0}^{\infty} e^{-2\tau} u(t-\tau+1) d\tau$$

The integrand in the above equation is nonzero only when $\tau \le t+1$. This suggests two regions of consideration: $t < -1$ and $-1 \le t$.

Case 1: $t < -1$. Here $u(t-\tau+1) = 0$, since $0 \le \tau$. Hence $f_6(t) = 0$, for $t < -1$.

Case 2: $-1 \le t$. Here $u(t-\tau+1)$ is nonzero only when $\tau \le t+1$. Therefore,

$$f_6(t) = 2\int_{0}^{t+1} e^{-2\tau} d\tau = e^{-2\tau}\Big]_{0}^{t+1} = 1 - e^{-2(t+1)}$$

It follows that

$$f_6(t) = \left[1 - e^{-2(t+1)}\right] u(t+1)$$

(d) By the definition of convolution and the sifting property of delta function we have

$$f_7(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_3(\tau - 2) d\tau = \int_{-\infty}^{\infty} \delta(t - \tau - 2) e^{-2(\tau-2)} u(\tau - 2) d\tau =$$

$$= e^{-2(\tau-2)} u(\tau - 2)\Big]_{\tau = t-2} = e^{-2(t-4)} u(t-4).$$

SOLUTION 16.6.

(a) By definition

$$f_6(t) = \left[1 - e^{-2(t+1)}\right] u(t+1)$$

Here observe that u(t – τ) = 0 for τ > t. Hence, there are two cases to consider: t ≤ 0 and t > 0.

Case 1**:** t ≤ 0

$$y(t) = K \int_{-\infty}^{t} e^{a\tau} d\tau = K \frac{e^{a\tau}}{a} \Bigg]_{-\infty}^{t} = K \frac{e^{at}}{a}$$

Case 2**:** t > 0

$$y(t) = K \int_{-\infty}^{t} e^{a\tau} d\tau = K \int_{-\infty}^{0} e^{a\tau} d\tau = K \frac{e^{a\tau}}{a} \Bigg]_{-\infty}^{0} = \frac{K}{a}$$

(b) By definition

$$y(t) = K \int_{-\infty}^{\infty} u(\tau - t) e^{-a\tau} u(\tau) d\tau = K \int_{t}^{\infty} u(\tau) e^{-a\tau} d\tau$$

Here observe that u(τ – t) = 0 for τ < t; hence the lower limit of integration is t. Also, because of the presence of u(τ) in the integrand, there are two cases to consider: t < 0 and t ≥ 0.

Case 1**:** t < 0

$$y(t) = K \int_{0}^{\infty} e^{-a\tau} d\tau = K \frac{e^{-a\tau}}{-a} \Bigg]_{0}^{\infty} = \frac{K}{a}$$

Case 2**:** t ≥ 0

$$y(t) = K \int_{t}^{\infty} e^{-a\tau} d\tau = K \frac{e^{-a\tau}}{-a} \Bigg]_{t}^{\infty} = \frac{K}{a} e^{-at}$$

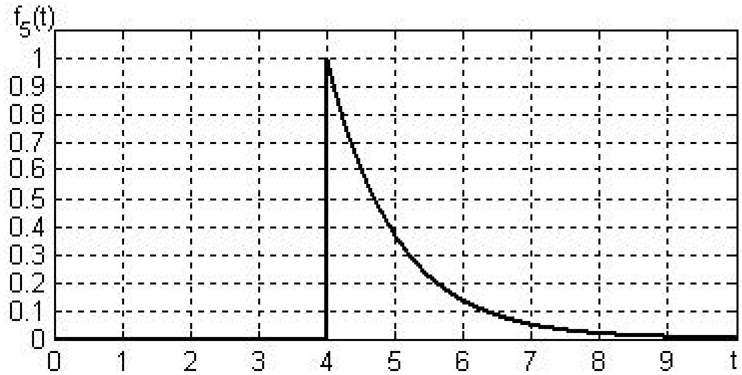
SOLUTION 16.7.

(a) Using the definition of the convolution integral and the sifting property of delta function, $f_5(t)$ can be computed as below

$$f_5(t) = \int_{-\infty}^{\infty} f_2(t-\tau) f_4(\tau) d\tau = \int_{-\infty}^{\infty} e^{-a(t-\tau)} u(t-\tau) \delta(\tau - 4) d\tau =$$

$$= e^{-a(t-\tau)} u(t-\tau)\Big]_{\tau=4} = e^{-a(t-4)} u(t-4)$$

A picture of $f_5(t)$, for $a = 1$, is sketched in the next figure.



(b) By definition

$$f_6(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_1(\tau) d\tau = K^2 \int_{-\infty}^{\infty} u(t-\tau)u(\tau)d\tau = K^2 \int_0^{\infty} u(t-\tau)d\tau$$

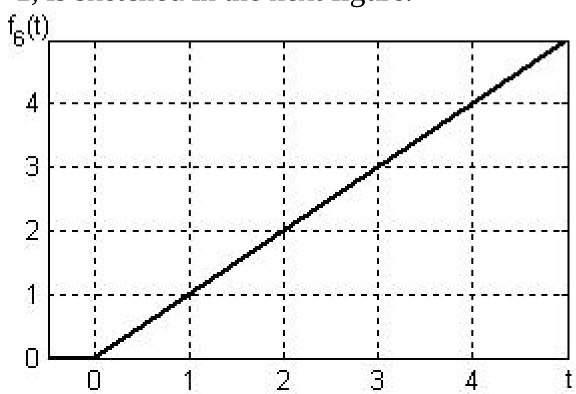
Since $u(t-\tau)$ is nonzero only when $\tau \le t$, there are two regions of consideration: $t < 0$ and $0 \le t$.
Case 1: $t < 0$. Here $u(t-\tau) = 0$, since $\tau \ge 0$. Hence

$$f_6(t) = 0, \text{ for } t < 0.$$

Case 2: $0 \le t$.

$$f_6(t) = K^2 \int_0^t d\tau = K^2 t, \text{ for } 0 \le t.$$

A picture of $f_6(t)$, for $K = 1$, is sketched in the next figure.



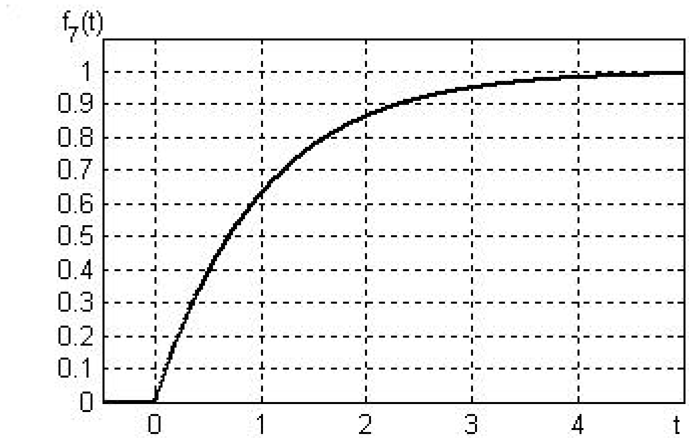
(c) By definition

$$f_7(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau = \int_{-\infty}^{\infty} Ku(t-\tau)e^{-a\tau}u(\tau)d\tau = K\int_0^{\infty} e^{-a\tau}u(t-\tau)d\tau$$

Since $u(t-\tau)$ is nonzero only when $\tau \le t$, there are two regions of consideration: $t < 0$ and $0 \le t$.
Case 1: $t < 0$. Here $u(t-\tau) = 0$, since $\tau \ge 0$. Therefore, $f_7(t) = 0$, for $t < 0$.
Case 2: $0 \le t$.

$$f_7(t) = K\int_0^t e^{-a\tau} d\tau = \frac{-K}{a} e^{-a\tau}\Big]_0^t = \frac{K}{a}\left(1 - e^{-at}\right), \text{ for } 0 \le t.$$

A picture of $f_7(t)$, for $t < 0$ and $a = 1$, is sketched in the next figure.



(d) By definition

$$f_8(t) = \int_{-\infty}^{\infty} f_1(t-\tau)\, f_3(\tau)\, d\tau = \int_{-\infty}^{\infty} Ku(t-\tau)e^{a\tau}\, u(-\tau)\, d\tau = K\int_{-\infty}^{0} e^{a\tau} u(t-\tau)\, d\tau$$

The integrand, $u(t-\tau)$, is nonzero only when $\tau \le t$. This suggests two regions of consideration: $t \le 0$ and $0 < t$.

Case 1: $t \le 0$.

$$f_8(t) = K\int_{-\infty}^{t} e^{a\tau} u(t-\tau)\, d\tau = \frac{K}{a} e^{a\tau}\Big]_{-\infty}^{t} = \frac{K}{a} e^{at}, \text{ for } t \le 0.$$

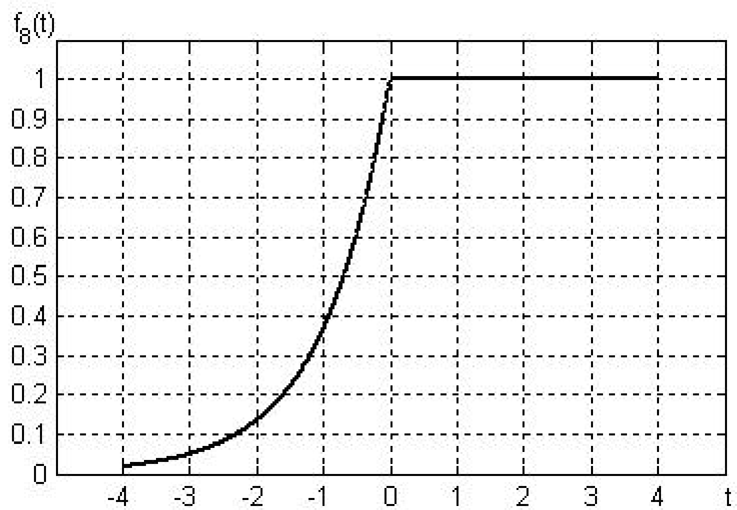
Case 2: $0 < t$.

$$f_8(t) = K\int_{-\infty}^{0} e^{a\tau} u(t-\tau)\, d\tau = \frac{K}{a} e^{a\tau}\Big]_{-\infty}^{0} = \frac{K}{a}, \text{ for } 0 < t.$$

In sum,

$$f_8(t) = \begin{cases} \dfrac{K}{a} e^{at}, & t \le 0 \\ \dfrac{K}{a}, & 0 < t \end{cases}$$

A picture of $f_8(t)$, for $K = 1$ and $a = 1$, is sketched in the next figure.



SOLUTION 16.8.

(a) Using the current division formula

$$I_c(s) = \frac{Cs}{Cs + \frac{1}{R}} I_{in}(s)$$

By Ohm's law the Laplace transform of capacitor's voltage

$$V_c(s) = \frac{1}{Cs} I_c(s)$$

Therefore the transfer function of the circuit

$$H(s) = \frac{V_c(s)}{I_{in}(s)} = \frac{1}{Cs + \frac{1}{R}} = \frac{1}{s+4}$$

Taking the inverse Laplace transform of $H(s)$ yields the impulse response $h(t) = e^{-4t}u(t)$.

(b) By the impulse response theorem

$$v_c(t) = i_{in}(t) \quad h(t) = \int_{-\infty}^{\infty} i_{in}(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} 3e^{-(t-\tau)}u(t-\tau)e^{-4\tau}u(\tau)d\tau =$$

$$= 3\int_{0}^{\infty} e^{-(t+3\tau)}u(t-\tau)d\tau$$

The integrand is nonzero only when $\tau \le t$. Therefore there are two regions of consideration: $t < 0$ and $0 \le t$.

Case 1: $t < 0$. Here $u(t-\tau) = 0$, since $0 \le \tau$. Hence $v_c(t) = 0$, for $t < 0$.

Case 2: $0 \le t$.

$$v_c(t) = 3\int_0^t e^{-(t+3\tau)}d\tau = -e^{-(t+3\tau)}\Big]_0^t = e^{-t} - e^{-4t}, \text{ for } 0 \le t.$$

In sum,

$$v_c(t) = \left(e^{-t} - e^{-4t}\right)u(t)\text{V}.$$

SOLUTION 16.9.

(a) By voltage division formula

$$V_{out}(s) = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_{in}(s)$$

Therefore the transfer function

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{1}{s+1}$$

Taking the inverse Laplace transform of $H(s)$ yields yields the impulse response $h(t) = e^{-t}u(t)$. By the impulse response theorem and the convolution definition

$$v_{out}(t) = \int_{-\infty}^{\infty} h(t-\tau)v_{in}(\tau)d\tau = \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)\left[u(-\tau) + 2e^{-2\tau}u(\tau)\right]d\tau =$$

$$= \int_{-\infty}^{0} e^{-(t-\tau)}u(t-\tau)d\tau + 2\int_0^{\infty} e^{-(t+\tau)}u(t-\tau)d\tau$$

For both integrals the integrand is nonzero only when $\tau \le t$. This suggests two regions of consideration: $t < 0$ and $0 \le t$.

Case 1: $t < 0$. Here the second integral is zero since, for this integral, τ is restricted to $[0,\infty)$.

$$v_c(t) = \int_{-\infty}^{t} e^{-(t-\tau)}d\tau = e^{\tau - t}\Big]_{-\infty}^{t} = 1, \text{ for } t < 0.$$

Case 2: $0 \le t$.

$$v_c(t) = \int_{-\infty}^{0} e^{-(t-\tau)}d\tau + 2\int_0^t e^{-(t+\tau)}d\tau$$

$$= e^{\tau - t}\Big]_{-\infty}^{0} - 2e^{-(t-\tau)}\Big]_0^t = e^{-t} - 2\left(e^{-2t} - e^{-t}\right) = 3e^{-t} - 2e^{-2t}, \text{ for } 0 \le t$$

(c) By the impulse response theorem and the definition of convolution

$$v_{out}(t) = \int_{-\infty}^{\infty} h(t-\tau)v_{in}(\tau)d\tau = \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)e^{-a|\tau|}d\tau$$

$$= \int_{-\infty}^{0} e^{-(t-\tau)}u(t-\tau)e^{a\tau}\,d\tau + \int_{0}^{\infty} e^{-(t-\tau)}u(t-\tau)e^{-a\tau}\,d\tau$$

$$= e^{-t}\int_{-\infty}^{0} e^{\tau(a+1)}u(t-\tau)d\tau + e^{-t}\int_{0}^{\infty} e^{\tau(1-a)}u(t-\tau)d\tau$$

For both integrals the integrand is nonzero only when $\tau \le t$. This suggests two regions of consideration: $t < 0$ and $0 \le t$.
Case 1: $t < 0$. Here the second integral is zero since, for this integral, τ is restricted to $[0,\infty)$.

$$v_{out}(t) = e^{-t}\int_{-\infty}^{t} e^{\tau(a+1)}d\tau = e^{-t}\left[\frac{e^{\tau(a+1)}}{a+1}\right]_{-\infty}^{t} = \frac{e^{at}}{a+1}, \text{ for } t < 0.$$

Case 2: $0 \le t$.

$$v_{out}(t) = e^{-t}\int_{-\infty}^{0} e^{\tau(a+1)}d\tau + e^{-t}\int_{0}^{t} e^{\tau(1-a)}d\tau =$$

$$= e^{-t}\left[\frac{e^{\tau(a+1)}}{a+1}\right]_{-\infty}^{0} + e^{-t}\int_{0}^{t} e^{\tau(1-a)}d\tau = \frac{e^{-t}}{a+1} + e^{-t}\int_{0}^{t} e^{\tau(1-a)}d\tau$$

Here observe that $a+1$ is nonzero since $a > 0$.

For computing the second integral, in case 2, we need to distinguish two subcases: $a = 1$ and $a \ne 1$.

$$e^{-t}\int_{0}^{t} e^{\tau(1-a)}d\tau = \begin{cases} e^{-t}t & \text{if } a = 1 \\ \dfrac{1}{1-a}\left(e^{-at} - e^{-t}\right) & \text{if } a \ne 1 \end{cases}$$

Therefore, for $0 \le t$,

$$v_{out}(t) = \begin{cases} \dfrac{e^{-t}}{a+1} + e^{-t}t & \text{if } a = 1 \\ \dfrac{e^{-t}}{a+1} + \dfrac{1}{1-a}\left(e^{-at} - e^{-t}\right) & \text{if } a \ne 1 \end{cases}$$

SOLUTION 16.10.
(a) The impulse response is obtained by taking the inverse Laplace transform of the transfer function

$$h(t) = -2e^{-0.2t}u(t)$$

By the impulse response theorem the response $y(t)$ equals

$$y(t) = h(t) \quad v(t) = \int_{-\infty}^{\infty} h(t-\tau)v(\tau)d\tau$$

Substituting $v(t) = u(t+1) - u(t-1)$ in the above integral yields

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)[u(\tau+1) - u(\tau-1)]d\tau$$

Here observe that $u(\tau+1) - u(\tau-1)$ is nonzero only when $-1 \le \tau \le 1$. Hence

$$y(t) = \int_{-1}^{1} h(t-\tau)d\tau = -2\int_{-1}^{1} e^{-0.2(t-\tau)}u(t-\tau)d\tau$$

The integrand is nonzero only when $\tau \le t$. This suggests three regions of consideration: $t < -1$, $-1 \le t < 1$ and $1 \le t$.
Case 1: $t < -1$. $y(t) = 0$
Case 2: $-1 \le t < 1$.

$$y(t) = -2\int_{-1}^{t} e^{-0.2(t-\tau)}d\tau = 10\left[e^{-0.2(t+1)} - 1\right], \text{ for } -1 \le t < 1.$$

Case 3: $1 \le t$.

$$y(t) = -2\int_{-1}^{1} e^{-0.2(t-\tau)}d\tau = 10e^{-0.2t}\left(e^{-0.2} - e^{0.2}\right), \text{ for } 1 \le t.$$

(b) The transfer function of the leaky integrator (see equation 14.14 in the textbook) is given by

$$H(s) = \frac{-\frac{1}{R_1}}{Cs + \frac{1}{R_2}}$$

where R_2 is the leakage resistance of the capacitor C and R_1 is the resistance connected at the inverting input of the op amp. Equating the two expressions of $H(s)$ we obtain that

$$\frac{-\frac{1}{R_1}}{Cs + \frac{1}{R_2}} = \frac{-2}{s + 0.2}$$

Matching the coefficients and taking into account that the smallest resistor is $10k\Omega$ the following values are obtained: $R_1 = 10k\Omega$, $R_2 = 100k\Omega$ and $C = 5 \ 10^{-5}F$.

(c) The impulse response is obtained by taking the inverse Laplace transform of the transfer function

$$h(t) = Ke^{-at}u(t)$$

By the impulse response theorem the response $y(t)$ equals

$$y(t) = h(t) \quad v(t) = \int_{-\infty}^{\infty} h(t-\tau)v(\tau)d\tau$$

Substituting $v(t) = u(t+T) - u(t-T)$ in the above integral yields

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\left[u(\tau + T) - u(\tau - T)\right]d\tau$$

Here observe that $u(\tau + T) - u(\tau - T)$ is nonzero only when $-T \le \tau \le T$. Hence

$$y(t) = \int_{-T}^{T} h(t-\tau)d\tau = K\int_{-T}^{T} e^{-a(t-\tau)}u(t-\tau)d\tau$$

The integrand is nonzero only when $\tau \le t$. This suggests three regions of consideration: $t < -T$, $-T \le t < T$ and $T \le t$.

Case 1: $t < -T$. $y(t) = 0$

Case 2: $-T \le t < T$.

$$y(t) = K\int_{-T}^{t} e^{-a(t-\tau)}d\tau = \frac{K}{a}\left[1 - e^{-a(t+T)}\right], \text{ for } -T \le t < T.$$

Case 3: $T \le t$.

$$y(t) = K\int_{-T}^{T} e^{-a(t-\tau)}d\tau = \frac{K}{a}e^{-at}\left(e^{aT} - e^{-aT}\right), \text{ for } T \le t.$$

In sum,

$$y(t) = \begin{cases} 0, & t < -T \\ \frac{K}{a}\left[1 - e^{-a(t+T)}\right], & -T \le t < T \\ \frac{K}{a}e^{-at}\left(e^{aT} - e^{-aT}\right), & T \le t \end{cases}$$

SOLUTION 16.11.

(a) First observe, from figure P16.11(a), that

$$f_2(t) = (-2t + 4)\left[u(t) - u(t-2)\right]$$

By definition

$$f_3(t) = \int_{-\infty}^{\infty} f_1(t-\tau)\, f_2(\tau)d\tau = \int_{-\infty}^{\infty} 4u(t-\tau)(-2\tau + 4)\left[u(\tau) - u(\tau - 2)\right]d\tau = -8\int_{0}^{2}(\tau - 2)u(t-\tau)d\tau$$

The integrand is nonzero only when $\tau \le t$. This suggests three regions of consideration: $t < 0$, $0 \le t < 2$ and $2 \le t$.

Case 1: $t < 0$. Here $u(t-\tau) = 0$ due to the fact that τ is restricted to the interval [0,2]. Hence

$$f_3(t) = 0, \text{ for } t < 0.$$

Case 2: $0 \le t < 2$.

$$f_3(t) = -8\int_0^t (\tau - 2)d\tau = -8\left[\frac{\tau^2}{2} - 2\tau\right]_0^t = -4\left(t^2 - 4t\right), \text{ for } 0 \le t < 2.$$

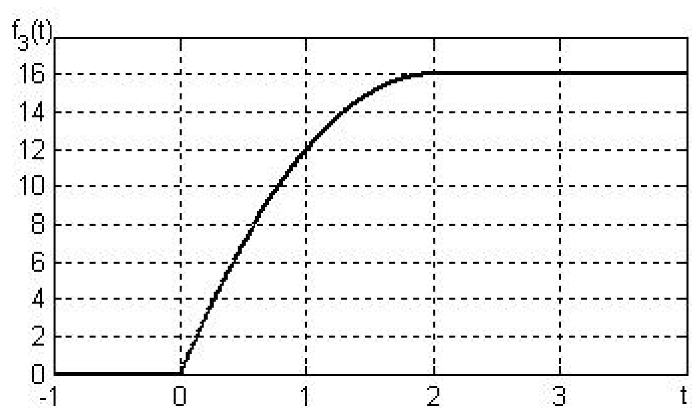
Case 3: $2 \le t$.

$$f_3(t) = -8\int_0^2 (\tau - 2)d\tau = -8\left[\frac{\tau^2}{2} - 2\tau\right]_0^2 = 16, \text{ for } 2 \le t.$$

In sum,

$$f_3(t) = \begin{cases} 0, & t < 0 \\ -4\left(t^2 - 4t\right), & 0 \le t < 2 \\ 16, & 2 \le t \end{cases}$$

A picture of $f_3(t)$ is sketched in the next figure.



(b) First observe, from figure P16.11(b), that

$$f_2(t) = t[u(t) - u(t-2)] + (4 - t)[u(t-2) - u(t-4)]$$

By definition

$$f_3(t) = \int_{-\infty}^{\infty} f_1(t-\tau)\, f_2(\tau)\, d\tau$$

By replacing $f_1(t)$ and $f_2(t)$ with their expressions we have

$$f_3(t) = \int_{-\infty}^{\infty} 4u(t-\tau)\{\tau[u(\tau) - u(\tau-2)] + (4-\tau)[u(\tau-2) - u(\tau-4)]\}d\tau =$$

$$= \int_{-\infty}^{\infty} 4u(t-\tau)\tau[u(\tau) - u(\tau-2)]d\tau + \int_{-\infty}^{\infty} 4u(t-\tau)(4-\tau)[u(\tau-2) - u(\tau-4)]d\tau =$$

$$= 4\int_0^2 \tau u(t-\tau)d\tau + 4\int_2^4 (4-\tau)u(t-\tau)d\tau$$

The integrands are nonzero only when $\tau \le t$. This suggests four regions of consideration: $t < 0$, $0 \le t < 2$, $2 \le t < 4$, and $4 \le t$.
Case 1: $t < 0$. Here $u(t-\tau) = 0$ due to the fact that τ is restricted to the interval $[0,4]$. Hence

$$f_3(t) = 0, \text{ for } t < 0.$$

Case 2: $0 \le t < 2$. here observe that the second integral is zero since, for this integral, τ is restricted to the interval $[2,4]$. Therefore

$$f_3(t) = 4\int_0^t \tau d\tau = 2t^2, \text{ for } 0 \le t < 2.$$

Case 3: $2 \le t < 4$.

$$f_3(t) = 4\int_0^2 \tau d\tau + 4\int_2^t (4-\tau)d\tau = 8 + 4\left[4\tau - \frac{\tau^2}{2}\right]_2^t = -2t^2 + 16t - 16, \text{ for } 2 \le t < 4.$$

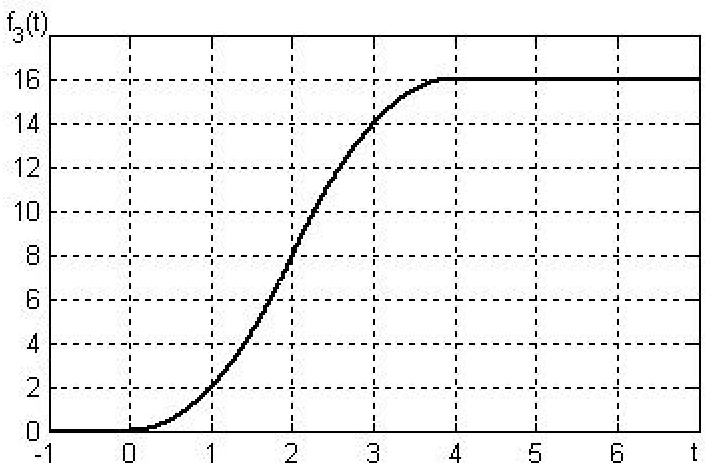
Case 4: $4 \le t$.

$$f_3(t) = 4\int_0^2 \tau d\tau + 4\int_2^4 (4-\tau)d\tau = 8 + 4\left[4\tau - \frac{\tau^2}{2}\right]_2^4 = 16, \text{ for } 4 \le t.$$

In sum,

$$f_3(t) = \begin{cases} 0, & t < 0 \\ 2t^2, & 0 \le t < 2 \\ -2t^2 + 16t - 16, & 2 \le t < 4 \\ 16, & 4 \le t \end{cases}$$

A picture of $f_3(t)$ is sketched in the next figure.



SOLUTION 16.12. (a) By voltage division,

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{\frac{2}{s}}{2s + 5 + \frac{2}{s}} = \frac{1}{s^2 + 2.5s + 1} = \frac{2/3}{(s + 0.5)} - \frac{2/3}{(s + 2)}$$

Hence, the impulse response is

$$h(t) = \frac{2}{3}e^{-0.5t}u(t) - \frac{2}{3}e^{-2t}u(t)$$

(b) By definition

$$h(t) * v_{in}(t) = \int_{-\infty}^{\infty} h(t-\tau)v_{in}(\tau)d\tau = \frac{20}{3}\int_{-\infty}^{\infty} e^{-0.5(t-\tau)}u(t-\tau)e^{\tau}u(-\tau)\,d\tau$$

$$- \frac{20}{3}\int_{-\infty}^{\infty} e^{-2(t-\tau)}u(t-\tau)e^{\tau}u(-\tau)\,d\tau$$

Case 1: t ≤ 0.

$$v_{out}(t) = \frac{20}{3}\int_{-\infty}^{t} e^{-0.5(t-\tau)}e^{\tau}\,d\tau - \frac{20}{3}\int_{-\infty}^{t} e^{-2(t-\tau)}e^{\tau}d\tau = \frac{20}{3}\int_{-\infty}^{t} e^{-0.5t}e^{1.5\tau}\,d\tau - \frac{20}{3}\int_{-\infty}^{t} e^{-2t}e^{3\tau}d\tau$$

$$= \frac{20e^{-0.5t}}{4.5}\left[e^{1.5\tau}\right]_{-\infty}^{t} - \frac{20e^{-2t}}{9}\left[e^{3\tau}\right]_{-\infty}^{t} = 4.444e^{t} - 2.222e^{t} = 2.222e^{t}$$

Case 2: t > 0.

$$v_{out}(t) = \frac{20e^{-0.5t}}{4.5}\left[e^{1.5\tau}\right]_{-\infty}^{0} - \frac{20e^{-2t}}{9}\left[e^{3\tau}\right]_{-\infty}^{0} = \frac{20e^{-0.5t}}{4.5} - \frac{20e^{-2t}}{9}$$

SOLUTION 16.13.
(a) The impulse response of the circuit has been computed in problem 16.12

$$h(t) = \frac{2}{3}e^{-0.5t}(t) - \frac{2}{3}e^{-2t}u(t)$$

By the impulse response theorem and the convolution definition

$$v_{out}(t) = h(t) \quad v_{in}(t) = \int_{-\infty}^{\infty} h(t-\tau)v_{in}(\tau)d\tau =$$

$$= \frac{20}{3}\int_{-\infty}^{\infty}\left[e^{-0.5(t-\tau)} - e^{-2(t-\tau)}\right]u(t-\tau)e^{-|\tau|}d\tau$$

The integrand is nonzero only when $\tau \le t$. Hence

$$v_{out}(t) = \frac{20}{3}\int_{-\infty}^{t}\left[e^{-0.5(t-\tau)} - e^{-2(t-\tau)}\right]e^{-|\tau|}d\tau$$

The existence of the function $e^{-|\tau|}$ under the integral suggests two regions of consideration: $t \le 0$ and $0 < t$.

Case 1: $t \le 0$.

$$v_{out}(t) = \frac{20}{3}\int_{-\infty}^{t}\left[e^{-0.5(t-\tau)} - e^{-2(t-\tau)}\right]e^{\tau}d\tau =$$

$$= \frac{20}{3}e^{-0.5t}\int_{-\infty}^{t}e^{1.5\tau}d\tau - \frac{20}{3}e^{-2t}\int_{-\infty}^{t}e^{3\tau}d\tau =$$

$$= \frac{20}{4.5}e^{-0.5t}\left[e^{1.5\tau}\right]_{-\infty}^{t} - \frac{20}{9}e^{-2t}\left[e^{3\tau}\right]_{-\infty}^{t} =$$

$$= 4.444e^{t} - 2.222e^{t} = 2.222e^{t}, \text{ for } t \le 0.$$

Case 2 : $0 < t$.

$$v_{out}(t) = \frac{20}{3}\int_{-\infty}^{0}\left[e^{-0.5(t-\tau)} - e^{-2(t-\tau)}\right]e^{\tau}d\tau + \frac{20}{3}\int_{0}^{t}\left[e^{-0.5(t-\tau)} - e^{-2(t-\tau)}\right]e^{-\tau}d\tau =$$

$$= \frac{20}{3}\int_{-\infty}^{0}\left[e^{-0.5t+1.5\tau} - e^{-2t+3\tau}\right]d\tau + \frac{20}{3}\int_{0}^{t}\left[e^{-0.5t-0.5\tau} - e^{-2t+\tau}\right]d\tau =$$

$$= \frac{20}{3}\left[\frac{e^{-0.5t+1.5\tau}}{1.5} - \frac{e^{-2t+3\tau}}{3}\right]_{-\infty}^{0} + \frac{20}{3}\left[\frac{e^{-0.5t-0.5\tau}}{-0.5} - e^{-2t+\tau}\right]_{0}^{t} =$$

$$= 17.778e^{-0.5t} - 20e^{-t} + 4.444e^{-2t}, \text{ for } 0 < t.$$

In sum,

$$v_{out}(t) = \begin{cases} 2.222e^{t}, & 0 \le t \\ 17.778e^{-0.5t} - 20e^{-t} + 4.444e^{-2t}, & 0 < t \end{cases}$$

SOLUTION 16.14.

(a) The impulse response of the circuit is obtained by taking the inverse Laplace transform of $H(s)$

$$h(t) = \left(2e^{-t} - 2e^{-2t} + 4e^{-4t}\right)u(t)$$

(b) The result follows from the following MATLAB code:

```
>> p = [-1,-2,-4];
>> r = [2,-2,4];
>> k = 0;
>> [n,d] = residue(r,p,k)
n =

     4    14    16


d =

     1     7    14     8
```

Therefore,

$$H(s) = \frac{4s^2 + 14s + 16}{s^3 + 7s^2 + 14s + 8}$$

(c) By the impulse response theorem

$$y(t) = u(t) \quad h(t) = u(t) * \left[2e^{-t}u(t) - 2e^{-2t}u(t) + 4e^{-4t}u(t)\right]$$

Using the distributive property of convolution we have

$$y(t) = u(t) \quad \left[2e^{-t}u(t)\right] + u(t) \quad \left[-2e^{-2t}u(t)\right] + u(t) \quad \left[4e^{-4t}u(t)\right]$$

In problem P16.2(c) the following equation has been obtained

$$\left[K_1 u(t)\right] \quad \left[K_2 e^{-at}u(t)\right] = \frac{K_1 K_2}{a}(1 - e^{-at})u(t)$$

Using the above equation $y(t)$ is immediately obtained

$$y(t) = 2\left(1 - e^{-t}\right)u(t) - \left(1 - e^{-2t}\right)u(t) + \left(1 - e^{-4t}\right)u(t) =$$

$$= \left(2 - 2e^{-t} + e^{-2t} - e^{-4t}\right)u(t)$$

(d) By the impulse response theorem

$$y(t) = f(t) \quad h(t) = \left[8u(-t) - 8u(-t - T)\right] * \left[2e^{-t}u(t) - 2e^{-2t}u(t) + 4e^{-4t}u(t)\right]$$

Using the distributive property of convolution we have

$$y(t) = 8u(-t) \quad \left[2e^{-t}u(t) - 2e^{-2t}u(t) + 4e^{-4t}u(t)\right] -$$
$$-8u(-t - T) * \left[2e^{-t}u(t) - 2e^{-2t}u(t) + 4e^{-4t}u(t)\right]$$

We denote

$$y_1(t) = 8u(-t) \quad \left[2e^{-t}u(t) - 2e^{-2t}u(t) + 4e^{-4t}u(t)\right]$$

By the time invariance property it follows that

$$y(t) = y_1(t) - y_1(t-T)$$

In order to compute $y_1(t)$ we will use the following equation which has been obtained in problem 16.2(e)

$$\left[K_1 u(-t)\right] \quad \left[K_2 e^{-at}u(t)\right] = \begin{cases} \dfrac{K_1K_2}{a}, & t \le 0 \\ \dfrac{K_1K_2}{a} e^{-at}, & t > 0 \end{cases}$$

Therefore,

$$y_1(t) = \begin{cases} 16, & t \le 0 \\ 16e^{-t} - 8e^{-2t} + 8e^{-4t}, & 0 < t \end{cases}$$

The zero-state response to the input $f(t)$ can now be computed

$$y(t) = \begin{cases} 0, & t \le 0 \\ 16e^{-t} - 8e^{-2t} + 8e^{-4t} - 16, & 0 < t \le T \\ 16\left[e^{-t} - e^{-(t-T)}\right] - 8\left[e^{-2t} - e^{-2(t-T)}\right] + 8\left[e^{-4t} - e^{-4(t-T)}\right], & T < t \end{cases}$$

SOLUTION 16.15.
(a) Using the convolution theorem the transfer function of the cascade is

$$H(s) = \mathrm{L}\left[h(t)\right] = \mathrm{L}\left[h_1(t) \quad h_2(t) \quad h_3(t)\right] = \mathrm{L}\left[h_1(t)\right] \; \mathrm{L}\left[h_2(t)\right] \; \mathrm{L}\left[h_3(t)\right] = H_1(s) \; H_2(s) \; H_3(s)$$

From table 13.1

$$H_1(s) = \frac{1}{s}$$

$$H_2(s) = \frac{10}{s+2}$$

$$H_3(s) = \frac{2}{s^2}$$

Therefore,

$$H(s) = \frac{20}{s^3(s+2)}$$

A partial fraction expansion of $H(s)$ can be obtained using the *residue* command in MATLAB:

```
>> num = [20];
>> den = [1 2 0 0 0];
>> [r,p,k] = residue(num,den)
```

```
r =

  -2.5000
   2.5000
  -5.0000
  10.0000

p =

   -2
    0
    0
    0

k =

    []
```

Hence

$$H(s) = \frac{-2.5}{s+2} + \frac{2.5}{s} + \frac{-5}{s^2} + \frac{10}{s^3}$$

Taking the inverse Laplace transform yields the impulse response of the cascade

$$h(t) = -2.5e^{-2t}u(t) + 2.5u(t) - 5tu(t) + 5t^2u(t)$$

(b) By the impulse response theorem and the convolution theorem, the Laplace transform of the step response of the cascade equals

$$Y(s) = H(s)\ U(s) = \frac{20}{s^3(s+2)}\ \frac{1}{s} = \frac{20}{s^4(s+2)}$$

A partial fraction expansion of $H(s)$ can be obtained using the *residue* command in MATLAB:

```
>> num = [20];
>> den = [1 2 0 0 0 0];
>> [r,p,k] = residue(num,den)

r =

   1.2500
  -1.2500
   2.5000
  -5.0000
  10.0000

p =

   -2
    0
    0
```

0
0

k =

[]

Hence

$$Y(s)=\frac{1.25}{s+2}+\frac{-1.25}{s}+\frac{2.5}{s^2}+\frac{-5}{s^3}+\frac{10}{s^4}$$

Taking the inverse Laplace transform yields the step response of the cascade

$$y(t)=1.25e^{-2t}u(t)-1.25u(t)+2.5tu(t)-2.5t^2u(t)+1.667t^3u(t)\,.$$

SOLUTION 16.16.
(a) By the voltage division formula

$$V_{out}(s)=\frac{\frac{1}{Cs}}{R+\frac{1}{Cs}}V_i(s)=\frac{1}{CRs+1}V_{in}(s)$$

Therefore, the transfer function of the circuit is

$$H(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{1}{CRs+1}=\frac{1}{s+1}$$

Taking the inverse Laplace transform yields the impulse response $h(t)=e^{-t}u(t)$.
(b) From table 13.1 the Laplace transform of $v_{in}(t)$ is

$$V_{in}(s)=\frac{1}{s}+\frac{1}{(s+1)^2}$$

By the impulse response theorem and the convolution theorem it follows that

$$V_{out}(s)=H(s)\;V_{in}(s)=\frac{1}{s+1}\left[\frac{1}{s}+\frac{1}{(s+1)^2}\right]=\frac{1}{s(s+1)}+\frac{1}{(s+1)^3}$$

A partial fraction expansion of $V_{out}(s)$ is

$$V_{out}(s)=\frac{1}{s}-\frac{1}{s+1}+\frac{1}{(s+1)^3}$$

Taking the inverse Laplace transform yields

$$v_{out}(t) = u(t) - e^{-t}u(t) - 0.5t^2e^{-t}u(t).$$

Using the time domain convolution method $v_{out}(t)$ can be computed as follows

$$v_{out}(t) = \int_{-\infty}^{\infty} h(t-\tau)v_{in}(\tau)d\tau =$$

$$= \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)\left[u(\tau) + \tau e^{-\tau}u(\tau)\right]d\tau$$

From the experience earned by computing convolution integrals we know that the computation of the above integral requires more computational work than the Laplace transform method. More computations imply, of course, more sources of errors.

From the solution of this problem we have seen that, in the case of the Laplace transform method, the computational burden consists in computing Laplace and inverse Laplace transforms. For a large class of functions these transforms can be found in tables(for example table 13.1). The only computation that we did, in the solution of this problem, was the partial fraction expansion of $V_{out}(s)$.

(c) In this case the (unilateral) transform method cannot be used because $v_{in}(t) \neq 0$ for $t < 0$.

SOLUTION 16.17.

(a) The impulse response can be obtained by taking the inverse Laplace transform of $H(s)$. Therefore

$$h(t) = 8e^{-10t}u(t)$$

(b) From table 13.1

$$V_{in}(s) = \frac{8}{s^2+16}$$

By the impulse response theorem and the convolution theorem it follows that

$$V_{out}(s) = H(s)\ V_{in}(s) = \frac{64}{(s+10)(s^2+16)}$$

$V_{out}(s)$ can be further written as

$$V_{out}(s) = \frac{-0.5517s + 5.517}{s^2+16} + \frac{0.5517}{s+10} = -0.5517\frac{s}{s^2+16} + 1.379\frac{4}{s^2+16} + \frac{0.5517}{s+10}$$

The above expansion of $V_{out}(s)$ can be obtained by using the technique of example 13.14, page 514. Taking the inverse Laplace transform yields

$$v_{out}(t) = -0.5517\cos(4t)u(t) + 1.379\sin(4t)u(t) + 0.5517e^{-10t}u(t).$$

(c) In this case

$$V_{in}(s) = \mathcal{L}\left[2e^{-2t}\sin(4t)u(t)\right] = \frac{8}{(s+2)^2+16}$$

Therefore

$$V_{out}(s) = H(s)\; V_{in}(s) = \frac{64}{\left[(s+2)^2+16\right](s+10)}$$

Using again the technique of example 13.14, page 514, $V_{out}(s)$ can be written as

$$V_{out}(s) = \frac{-0.8s+4.8}{(s+2)^2+16} + \frac{0.8}{s+10} = -0.8\frac{s+2}{(s+2)^2+16} + 1.6\frac{4}{(s+2)^2+16} + \frac{0.8}{s+10}$$

Taking the inverse Laplace transform yields

$$v_{out}(t) = -0.8e^{-2t}\cos(4t)u(t) + 1.6e^{-2t}\sin(4t)u(t) + 0.8e^{-10t}u(t) \text{ V.}$$

In this context the Laplace transform method is faster than the time domain convolution. This is due to the fact that $v_{in}(t)$ has a relatively complicated expression and the convolution integral will require more computational work than the Laplace transform method.

(d) In this context the Laplace transform method cannot be used because $v_{in}(t) \neq 0$ for $t < 0$. The time domain convolution method will be used to compute the response $v_{out}(t)$. By the impulse response thorem

$$v_{out}(t) = h(t) \quad v_{in}(t) = \left[8e^{-10t}u(t)\right] \quad [u(-t)]$$

Using the result of problem 16.2, part (e), it follows that

$$v_{out}(t) = \begin{cases} 0.8, & t \le 0 \\ 0.8e^{-10t}, & 0 < t \end{cases}$$

SOLUTION 16.18.

(a) Replacing R_1, R_2, C_1 and C_2 with their values the transfer function can be obtained

$$H(s) = \frac{s}{s^2+5s+2}$$

The only zero of $H(s)$ is 0 and the poles of $H(s)$ are –0.5 and –2. A partial fraction expansion of $H(s)$ is:

$$H(s) = \frac{-0.167}{s+0.5} + \frac{0.667}{s+2}$$

The impulse response can be obtained by taking the inverse Laplace transform of $H(s)$

$$h(t) = -0.167e^{-0.5t} + 0.667e^{-2t}$$

(b) $v_{out}(t)$ will be computed using the Laplace transform method. This approach is valid because $h(t)$ and $v_{in}(t)$ are zero for $t<0$.
From table 13.1 the Lapace transform of $v_{in}(t)$ is

$$V_{in}(s)=\frac{1}{(s+2)^2}$$

By the impulse response theorem and the convolution theorem it follows that

$$V_{out}(s)=H(s)\ V(s)=\frac{s}{2s^2+5s+2}\frac{1}{(s+2)^2}=\frac{s}{2s^4+13s^3+30s^2+28s+8}$$

A partial fraction expansion of $V_{out}(s)$ can be obtained using the *residue* command in MATLAB:

```
>> a = [1 0];
>> b = [2 13 30 28 8];
>> [r,p,k] = residue(a,b)

r =
    0.0741
    0.1111
    0.6667
   -0.0741

p =
   -2.0000
   -2.0000
   -2.0000
   -0.5000

k =
     []
```

Therefore,

$$V_{out}(s)=\frac{-0.0741}{s+0.5}+\frac{0.0741}{s+2}+\frac{0.1111}{(s+2)^2}+\frac{0.6667}{(s+2)^3}$$

Taking the inverse Laplace transform yields

$$v_{out}(t)=\left[-0.0741e^{-0.5t}+0.0741e^{-2t}+0.1111te^{-2t}+0.3333t^2e^{-2t}\right]u(t)\text{ V}.$$

One would prefer the time domain convolution method to compute $v_{out}(t)$, but the computations may require more work relatively to the Laplace transform method.

(c) In this part $v_{in}(t)\neq 0$ for $t<0$. Therefore the time domain convolution method will be used to compute $v_{out}(t)$.
By the impulse response theorem

$$v_{out}(t)=h(t)\quad v_{in}(t)=$$

$$=\int_{-\infty}^{\infty}\left[-0.1667e^{-0.5(t-\tau)}+0.6667e^{-2(t-\tau)}\right]u(t-\tau)e^{2\tau}u(-\tau)d\tau=$$

$$= \int_{-\infty}^{0} \left[-0.1667 e^{-0.5(t-\tau)} + 0.6667 e^{-2(t-\tau)} \right] u(t-\tau) e^{2\tau} \, d\tau$$

The integrand of the previous integral is nonzero only when $\tau \le t$. This suggests two regions of consideration: $t < 0$ and $0 \le t$.

Case 1: $t < 0$.

$$v_{out}(t) = -0.1667 e^{-0.5t} \int_{-\infty}^{t} e^{2.5\tau} \, d\tau + 0.6667 e^{-2t} \int_{-\infty}^{t} e^{4\tau} d\tau =$$

$$= -0.1667 e^{-0.5t} \left[\frac{e^{2.5\tau}}{2.5} \right]_{-\infty}^{t} + 0.6667 e^{-2t} \left[\frac{e^{4\tau}}{4} \right]_{-\infty}^{t} =$$

$$= 0.1 e^{-2t}, \text{ for } t < 0.$$

Case 2: $0 \le t$.

$$v_{out}(t) = -0.1667 e^{-0.5t} \int_{-\infty}^{0} e^{2.5\tau} \, d\tau + 0.6667 e^{-2t} \int_{-\infty}^{0} e^{4\tau} d\tau =$$

$$= -0.1667 e^{-0.5t} \left[\frac{e^{2.5\tau}}{2.5} \right]_{-\infty}^{0} + 0.6667 e^{-2t} \left[\frac{e^{4\tau}}{4} \right]_{-\infty}^{0} =$$

$$= -0.1667 e^{-0.5t} + 0.6667 e^{-2t}, \text{ for } 0 \le t.$$

SOLUTION 16.19.

Replacing R and C with their values

$$H(s) = \frac{s-5}{s+5} = 1 - \frac{10}{s+5}$$

The zero-state response $v_{out}(t)$ will be computed using the time domain convolution method because $v_{in}(t) \ne 0$ for $t < 0$.

The impulse response of the circuit is

$$h(t) = \delta(t) - 10 e^{-5t} u(t)$$

By the impulse response theorem

$$v_{out}(t) = h(t) \quad v_{in}(t) = \int_{-\infty}^{\infty} h(\tau) v_{in}(t-\tau) d\tau =$$

$$= 10 \int_{-\infty}^{\infty} \left[\delta(\tau) - 10 e^{-5\tau} u(\tau) \right] cos[10(t-\tau)] d\tau =$$

$$= 10 \int_{-\infty}^{\infty} \delta(\tau) cos(t-\tau) d\tau - 10 \int_{-\infty}^{\infty} 10 e^{-5\tau} u(\tau) cos[10(t-\tau)] d\tau$$

Using the sifting property of the delta function and expanding $\cos(t-\tau)$ it follows that

$$v_{out}(t) = 10\cos(t-\tau) - 100\cos(10t)\int_0^\infty e^{-5\tau}\cos(10\tau)\,d\tau - 100\sin(10t)\int_0^\infty e^{-5\tau}\sin(10\tau)\,d\tau$$

Using the definition of the Laplace transform we observe that

$$\int_0^\infty e^{-5\tau}\cos(10\tau)\,d\tau = \mathcal{L}\left[\cos(10t)u(t)\right]_{s=5} = \left.\frac{s}{s^2+100}\right|_{s=5} = 0.04$$

and

$$\int_0^\infty e^{-5\tau}\sin(10\tau)\,d\tau = \mathcal{L}\left[\sin(10t)u(t)\right]_{s=5} = \left.\frac{10}{s^2+100}\right|_{s=5} = 0.08$$

Therefore

$$\begin{aligned} v_{out}(t) &= 10\cos(10t) - 4\cos(10t) - 8\sin(10t) \\ &= 6\cos(t) - 8\sin(10t) = 10\cos\left[10t + \tan^{-1}(\tfrac{4}{3})\right] \end{aligned}$$

Notice that $v_{out}(t)$ and $v_{in}(t)$ have the same frequency and magnitude.

SOLUTION 16.20.

(a) First notice that $v_{in}(t-T) = u(t)$. Therefore $w(t) = u(t)$ and

$$W(s) = \frac{1}{s}$$

(b) Using the properties of the Laplace transform it follows that

$$V_{out}^w(s) = H(s)\;W(s) = \frac{2}{s(s+2)}$$

A partial fraction expansion of $V_{out}^w(s)$ is

$$V_{out}^w(s) = \frac{1}{s} - \frac{1}{s+2}$$

Taking the inverse Laplace transform yields

$$v_{out}^w(t) = u(t) - e^{-2t}u(t)$$

(b) Since

$$v_{in}(t) = w(t+T)$$

it follows, by the time invariance property, that

$$v_{out}^v(t) = v_{out}^w(t+T).$$

Therefore,

$$v_{out}^v(t) = u(t+T) - e^{-2(t+T)}u(t+T)\ \text{V}.$$

SOLUTION 16.21.

(a) First observe from figure P16.21 that

$$v_{in}(t) = u(t+T) - u(t-T)$$

From the definition of $w(t)$ it follows that

$$w(t) = v_{in}(t - T) = u(t) - u(t - 2T)$$

Therefore

$$W(s) = \frac{1}{s} - \frac{1}{s} e^{-2sT}$$

(a) By the impulse response theorem and the convolution theorem it follows that

$$V_{out}^{w}(s) = H(s)\ W(s) = \frac{2}{s(s+2)}\left(1 - e^{-2sT}\right) = \left(\frac{1}{s} - \frac{1}{s+2}\right)\left(1 - e^{-2sT}\right) =$$

$$= \frac{1}{s} - \frac{1}{s+2} - \left(\frac{1}{s} - \frac{1}{s+2}\right) e^{-2sT}$$

Taking the inverse Laplace transform and using the time shift property of the Laplace transform yields

$$v_{out}^{w}(t) = \left(1 - e^{-2t}\right)u(t) - \left[1 - e^{-2(t-2T)}\right]u(t - 2T)$$

Because

$$v_{in}(t) = w(t + T)$$

it follows, by the time invariance property, that

$$v_{out}^{v}(t) = v_{out}^{w}(t + T) =$$

$$= \left(1 - e^{-2(t+T)}\right)u(t + T) - \left[1 - e^{-2(t-T)}\right]u(t - T)\ \text{V.}$$

SOLUTION 16.22. (a) The use of $t = t + T_1$ in the problem statement means replace t by $t + T_1$. However, strictly speaking we should have used a statement of the form $t = t' + T_1$ which is done in the proof below. By definition of the convolution and the property of commutivity,

$$f(t - T_1) * g(t) = \int_{-\infty}^{\infty} f(t - T_1 - \tau) g(\tau)\, d\tau = \left[\int_{-\infty}^{\infty} f(t' - \tau) g(\tau)\, d\tau\right]_{t' = t - T_1} = \left[f(t') * g(t')\right]_{t' = t - T_1}$$

Observe that $t = t' + T_1$. Hence

$$\left[f(t - T_1) * g(t)\right]_{t = t' + T_1} = f(t') * g(t')$$

Realizing that t and t' are simply dummy variables, we immediately obtain the result. From a systems perspective, this corresponds to the property of time-invariance where a shift of an input function by T_1 yields a corresponding shift of the output function by T_1.

(b) The steps in this part are similar to those of part (a).

$$f(t-T_1)*g(t-T_2) = \int_{-\infty}^{\infty} f(t-T_1-\tau)g(\tau-T_2)\,d\tau$$

Let $\lambda = \tau - T_2$ in which case $\tau = \lambda + T_2$ and $d\lambda = d\tau$. Hence

$$f(t-T_1)*g(t-T_2) = \int_{-\infty}^{\infty} f(t-T_1-T_2-\lambda)g(\lambda)\,d\lambda = \left[\int_{-\infty}^{\infty} f(t'-\lambda)g(\lambda)\,d\lambda\right]_{t'=t-T_1-T_2}$$

$$= \left[f(t')*g(t')\right]_{t'=t-T_1-T_2}$$

Since $t = t'+T_1+T_2$, and t and t' are dummy variables, we have

$$\left[f(t-T_1)*g(t-T_2)\right]_{t=t'+T_1+T_2} = f(t')*g(t')$$

and the result follows.

SOLUTION 16.23. From table 13.1, $G(s) = \frac{1}{s+2}$, $H(s) = \frac{1}{(s+2)^2}$.

(a) Define p(t) = f(t – 2) = u(t).

Then, $P(s) = \frac{1}{s}$. Consider

$$P(s)G(s) = \frac{1}{s(s+2)} = \frac{0.5}{s} - \frac{0.5}{s+2}$$

Hence

$$p(t)*g(t) = 0.5u(t) - 0.5e^{-2t}u(t)$$

From problem 16.22 part (a),

$$f(t)*g(t) = \left[p(t)*g(t)\right]_{t=t+2} = \left[0.5u(t) - 0.5e^{-2t}u(t)\right]_{t=t+2} = \left[0.5 - 0.5e^{-2(t+2)}\right]u(t+2)$$

(b) Define p(t) = f(t – 2) = u(t).

Then, $P(s) = \frac{1}{s}$. Consider

$$P(s)H(s) = \frac{1}{s(s+2)^2} = \frac{0.25}{s} - \frac{0.25}{s+2} - \frac{0.5}{(s+2)^2}$$

Hence

$$p(t) * h(t) = \left(0.25 - 0.25e^{-2t} - 0.5te^{-2t}\right)u(t)$$

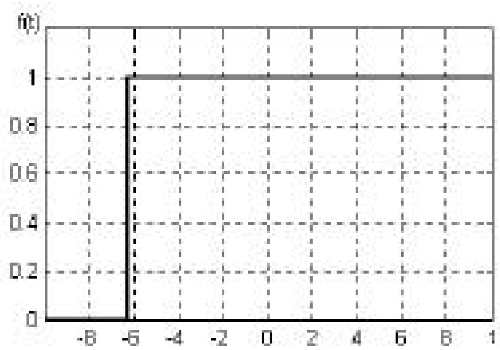
From problem 16.22 part (a),

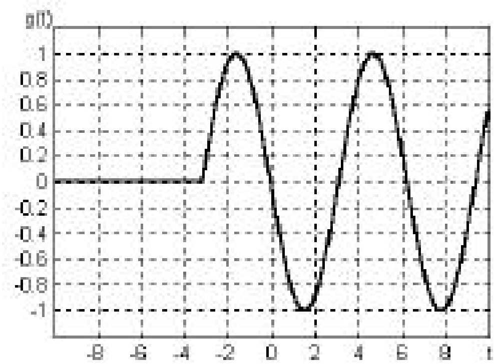
$$f(t) * h(t) = \left[p(t) * h(t)\right]_{t=t+2} = \left[\left(0.25 - 0.25e^{-2t} - 0.5te^{-2t}\right)u(t)\right]_{t=t+2}$$

$$= \left[0.25 - 0.25e^{-2(t+2)} - 0.5(t+2)e^{-2(t+2)}\right]u(t+2)$$

SOLUTION 16.24.

(a) The pictures of $f(t)$ and $g(t)$ are sketched in the next figures





(b) Using the convolution theorem it follows that

$$\mathcal{L}\left[f(t-2\pi) \quad g(t-\pi)\right] = \mathcal{L}\left[f(t-2\pi)\right] \; \mathcal{L}\left[g(t-\pi)\right]$$

From table 13.1

$$\mathcal{L}\left[f(t-2\pi)\right] = \mathcal{L}\left[u(t)\right] = \frac{1}{s}$$

$$\mathcal{L}\left[g(t-\pi)\right] = \mathcal{L}\left[\sin(t)u(t)\right] = \frac{1}{s^2+1}$$

Therefore

$$\mathcal{L}\left[f(t-2\pi) \quad g(t-\pi)\right] = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

Taking the inverse Laplace transform yields

$$f(t-2\pi) \quad g(t-\pi) = u(t) - \cos(t)u(t) = \left[1-\cos(t)\right]u(t)$$

Using the property give in problem 16.22(b) it follows that

$$f(t) \quad g(t) = \left[f(t-2\pi) \quad g(t-\pi)\right]_{t=t+2\pi+\pi} = \left[1-\cos(t+3\pi)\right]u(t+3\pi)$$

SOLUTION 16.25.
Define

$$w(t) = v_{in}(t-2)$$

Hence

$$w(t) = u(t)$$

and, from table 13.1,

$$W(s) = \frac{1}{s}$$

From table 13.1 we also have that

$$H_1(s) = \frac{1}{s+1} \text{ and } H_2(s) = \frac{1}{(s+1)^2}$$

The impulse response of the cascade is

$$h(t) = h_1(t) \quad h_2(t)$$

Hence the transfer function of the cascade is

$$H(s) = H_1(s)H_2(s) = \frac{1}{(s+1)^3}$$

We denote by $v_{out}^w(t)$ the zero state response due to the input $w(t)$. Hence,

$$V_{out}^w(s) = H(s)\ \ W(s) = \frac{1}{s(s+1)^3}$$

A partial fraction expansion of $V_{out}^w(s)$ is obtained using the *residue* command in MATLAB:

```
 >> a = [1];
>> b = [1 3 3 1 0];
>> [r,p,k] = residue(a,b)

r =
   -1.0000
   -1.0000
   -1.0000
    1.0000

p =
   -1.0000
   -1.0000
   -1.0000
         0

k =
     []
```

Therefore

$$V_{out}^w(s) = \frac{1}{s} + \frac{-1}{s+1} + \frac{-1}{(s+1)^2} + \frac{-1}{(s+1)^3}$$

Taking the inverse Laplace transform yields

$$v_{out}^{w}(t) = \left[1 - e^{-t} - te^{-t} - 0.5t^2 e^{-t}\right]u(t) \text{ V}$$

Due to the fact that

$$v_{in}(t) = w(t+2)$$

the time invariance property implies that

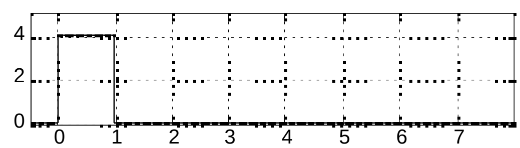
$$v_{out}(t) = v_{out}^{w}(t+2) =$$
$$= \left[1 - e^{-(t+2)} - (t+2)e^{-(t+2)} - 0.5(t+2)^2 e^{-(t+2)}\right]u(t+2) \text{ V.}$$

SOLUTION 16.26.

(a) Using the sifting property of the delta function it follows that

$$f_4(t) = [\delta(t) + \delta(t-4)] \quad f_2(t) = f_2(t) + f_2(t-4)$$

The right-hand side of the above equation interprets as a graphical sum of shifted pictures of $f_2(t)$. A picture of $f_4(t)$ is sketched in the next figure.



(b) In order to compute the area beneath $f_2(t-\tau) \quad f_2(\tau)$ four regions will be considered: $t < 0$, $0 \le t < 1, 1 \le t < 2$ and $2 \le t$.

Step 1: $t < 0$. In this case $f_2(t-\tau) \quad f_2(\tau) = 0$ for all τ. Therefore

$$f_2(t) \quad f_2(t) = 0 \text{ for } t < 0.$$

Step 2: $0 \le t < 1$. In this case $f_2(t-\tau) \quad f_2(\tau) = 16$ for $0 \le \tau \le t$ and is zero elsewhere. The area beneath $f_2(t-\tau) \quad f_2(\tau)$ equals $16t$. Therefore

$$f_2(t) \quad f_2(t) = 16t \text{ for } 0 \le t < 1.$$

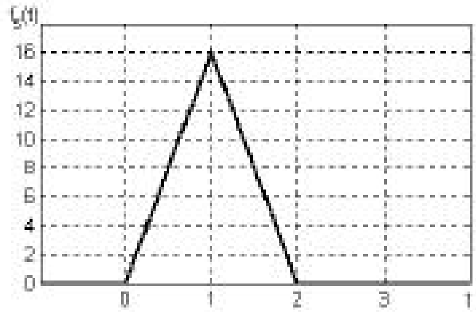
Step 3: $1 \le t < 2$. In this case $f_2(t-\tau) \quad f_2(\tau) = 16$ for $t-1 < \tau \le 1$ and is zero elsewhere. Hence the area beneath $f_2(t-\tau) \quad f_2(\tau)$ equals

$$f_2(t) \quad f_2(t) = 16(2-t) \text{ for } 1 \le t < 2.$$

Step 4: $2 \le t$. In this case $f_2(t-\tau) \quad f_2(\tau) = 0$ for all τ. Therefore

$$f_2(t) \quad f_2(t) = 0 \text{ for } 2 \le t.$$

A picture of $f_5(t)$ is sketched in the next figure.



(c) In order to compute the area beneath $f_2(t-\tau) \quad f_3(\tau)$ five regions will be considered: $t<0$, $0 \le t < 1$, $1 \le t < 2$, $2 \le t < 3$ and $3 \le t$.

Step 1: $t<0$. In this case $f_2(t-\tau) \quad f_3(\tau)=0$ for all τ. Therefore $f_2(t) \quad f_3(t)=0$ for $t<0$.

Step 2 : $0 \le t < 1$. In this case $f_2(t-\tau) \quad f_3(\tau)=8$ for $0 \le \tau \le t$ and is zero elsewhere. Therefore the area beneath $f_2(t-\tau) \quad f_3(\tau)$ equals

$$f_2(t) \quad f_3(\tau) = 8t \text{ for } 0 \le t < 1.$$

Step 3: $1 \le t < 2$. Here

$$f_2(t-\tau) \quad f_3(\tau) = \begin{cases} 8, & t-1<\tau<1 \\ 24, & 1 \le \tau < t \\ 0, & \text{otherwise} \end{cases}$$

Hence, the area beneath $f_2(t-\tau) \quad f_3(\tau)$ equals

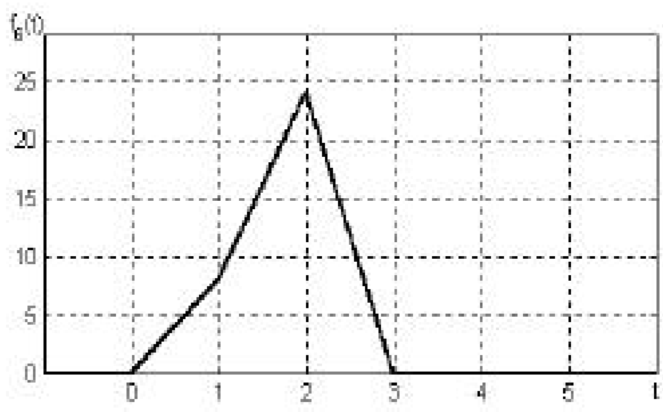
$$f_2(t) \quad f_3(\tau) = 8[1-(t-1)] + 24(t-1) = 8(2t-1) \text{ for } 1 \le t < 2.$$

Step 4: $2 \le t < 3$. In this case $f_2(t-\tau) \quad f_3(\tau)=24$ for $t-1<\tau<2$ and is zero otherwise. Hence, the area beneath $f_2(t-\tau) \quad f_3(\tau)$ equals

$$f_2(t) \quad f_3(t) = 24[2-(t-1)] = 24(3-t) \text{ for } 2 \le t < 3.$$

Step 5: $3 \le t$. Here $f_2(t-\tau) \quad f_3(\tau)=0$ for all τ. Therefore $f_2(t) \quad f_3(t)=0$ for $t<0$.

A picture of $f_6(t)$ is sketched in the next figure.



SOLUTION 16.27.

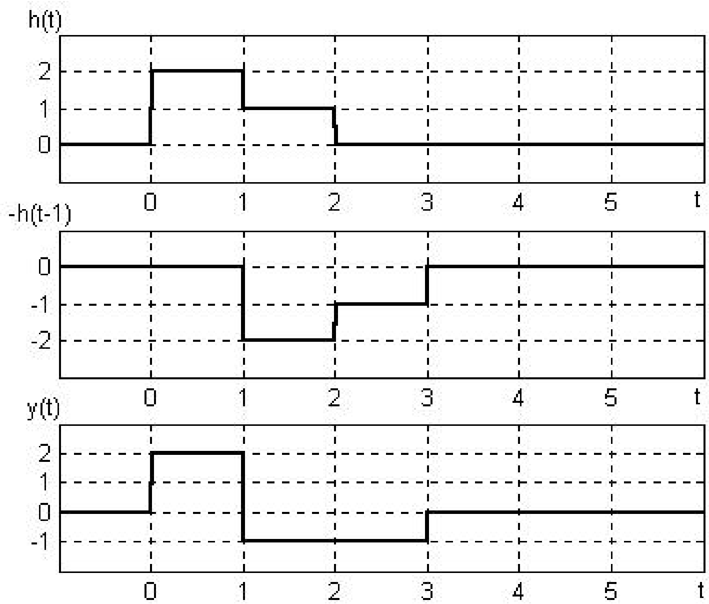
By the impulse response theorem, it follows that the response is

$$y(t) = h(t) \quad f(t) =$$
$$= h(t) \quad \left[\delta(t) - \delta(t-1)\right]$$

Using the distributive property of convolution and the sifting property of delta function $y(t)$ can be written as

$$y(t) = h(t) - h(t-1)$$

The right-hand side of the above equation interprets as a graphical sum of (shifted) pictures of $h(t)$. The pictures of $h(t)$, $h(t-1)$ and $y(t)$ are sketched in the next figures.



SOLUTION 16.28.

(a) From the picture of $f(t)$ and $h(t)$ in figure P16.28 we observe that, in order to compute the area beneath $h(t-\tau) f(\tau)$, we need to consider four cases: $t < 0$, $0 \le t < 4$, $4 \le t < 8$ and $8 \le t$.

Step 1: $t < 0$. Here $h(t-\tau) f(\tau) = 0$ for all τ. Therefore the area beneath $h(t-\tau) f(\tau)$ equals zero and

$$h(t) \quad f(t) = 0 \text{ for } t < 0.$$

Step 2: $0 \le t < 4$. In this case $h(t-\tau) f(\tau) = 1$ for $0 \le \tau \le t$ and is zero otherwise. Hence the area beneath $h(t-\tau) f(\tau)$ equals

$$h(t) \quad f(t) = t \text{ for } 0 \le t < 4.$$

Step 3: $4 \le t < 8$. In this case

$$h(t-\tau) f(\tau) = \begin{cases} 1, & t-4 < \tau \le 4 \\ 2, & 4 \le \tau \le t \\ 0, & \text{otherwise} \end{cases}$$

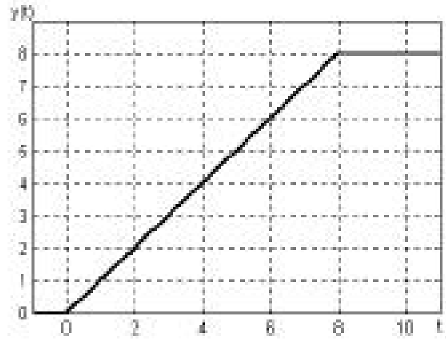
Therefore the area beneath $h(t-\tau) f(\tau)$ equals

$$h(t) \quad f(t) = \left[4 - (t-4)\right] + 2(t-4) = t \text{ for } 4 \le t < 8.$$

Step 4: $8 \le t$. Here $h(t-\tau)f(\tau) = 2$ for $t-4 < \tau \le t$ and is zero otherwise. Hence

$$h(t) \quad f(t) = 2\big[t-(t-4) = 8\big] \text{ for } 8 \le t.$$

A picture of $y(t)$ is sketched in the next figure.



(b) The impulse response is

$$h(t) = u(t) - u(t-4)$$

By the impulse response theorem

$$y(t) = x(t) \quad h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\big[u(t-\tau) - u(t-\tau-4)\big]d\tau$$

Here observe that $u(t-\tau) - u(t-\tau-4)$ is nonzero only when $t \le \tau < t-4$.
Therefore

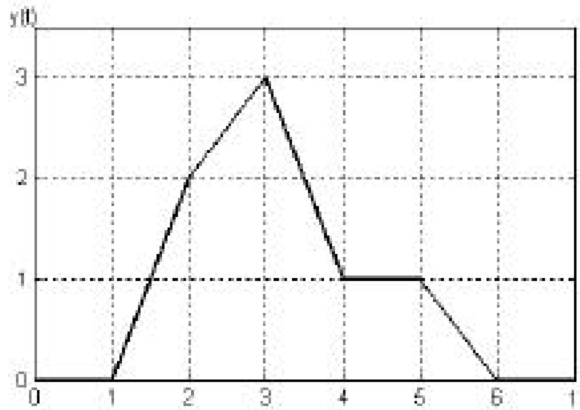
$$y(t) = \int_{t}^{t-4} x(\tau)d\tau$$

which interprets as the running area under $x(t)$ over the interval $[t-4, t]$.

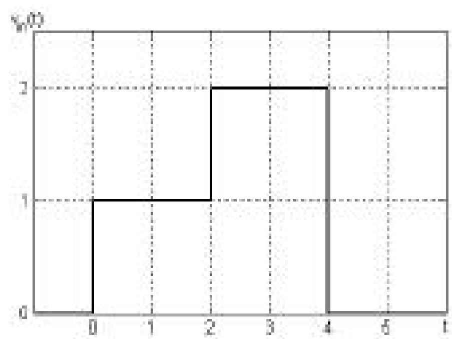
SOLUTION 16.29.
The response, $y(t)$, is obtained as indicated in the statement of the problem, by using the following MATLAB code:

```
>> tstep = 1;
>> vin = [1];
>> h = [0, 2, 3, 1, 1];
>> y = tstep*conv(vin, h);
>> y = [0 y 0];
>> t = 0:tstep:tstep*(length(vin)+length(h));
>> plot(t,y)
>> grid
```

The response is plotted in the next figure.



SOLUTION 16.30.
A picture of $v_{in}(t)$ sketched in the next figure.



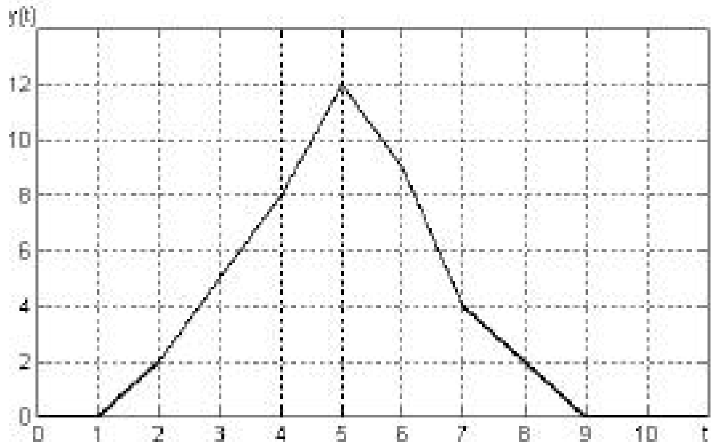
In order to plot the response, $y(t)$, the MATLAB code of problem 16.29 will be used with only one modification. Namely

vin = [1, 1, 2, 2]

as it can be observed from the picture of $v_{in}(t)$ with the time step tstep = 1.
Therefore the MATLAB code is:

```
>> tstep = 1;
>> vin = [1, 1, 2, 2];
>> h = [0, 2, 3, 1, 1];
>> y = tstep*conv(vin,h);
>> y = [0 y 0];
>> t = 0:tstep:tstep*(length(vin)+length(h));
>> plot(t,y)
>> grid
```

The response is plotted in the next figure.


y(t)
12
10
8
6
4
2
0
0
1
2
3
4
5
6
7
8
9
10
t