Functions And Linear Transformations

Functions

A function is a mathematical relationship that uniquely associates elements of one set (called the domain) with elements of another set (called the codomain). In simpler terms, a function maps inputs to outputs in a specific way.

Notation : A function f mapping elements from set X (domain) to set Y (codomain) is denoted by $f: X \longrightarrow Y$

If x is an element of X, then $f(x)$ is the corresponding element in Y.

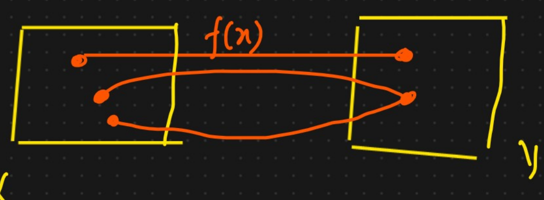
Example : $f(x) = 2x + 3 \Rightarrow$ maps each real number x to a real Number

$2x+3$

$x \xrightarrow{f:} y$.

$x = 5$

$f(2) = 2 \times 2 + 3 = 7 \Rightarrow f(x) \rightarrow$ Mapping $2 \in R$ to $7 \in R$



$g(x) \quad f: X \longrightarrow Y$

$f(x) = \begin{bmatrix} x+y \\ 6z \end{bmatrix}$

$f: \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3 \xrightarrow{f(x)} \begin{bmatrix} x+y \\ 6z \end{bmatrix} \in R^2 \Rightarrow$ 3 dimension $\Rightarrow$ 2 dimension vector

$f: \underbrace{R^3}_{\text{Domain}} \longrightarrow \underbrace{R^2}_{\text{Codomain.}}$

$f(x) \Rightarrow$ Transformation.

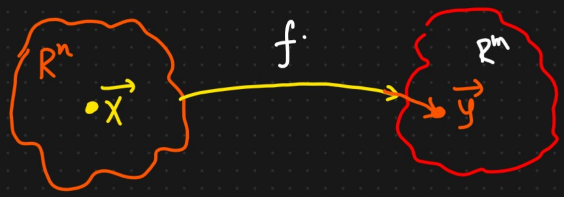
Eg : Dimensionality Reduction

Vector Transformations

$f : X \to Y$

$\vec{x} \to \vec{y}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad x_1, x_2, x_3 \cdots x_n \in R$$

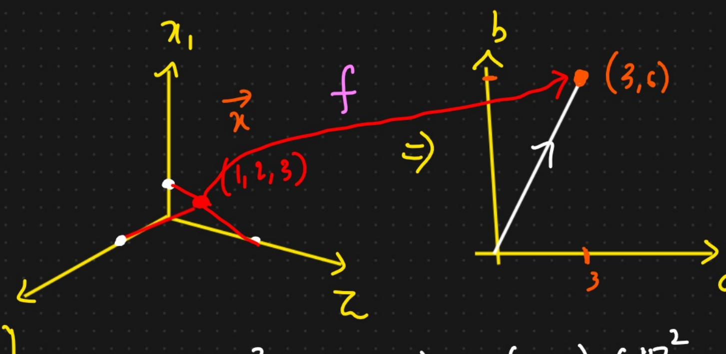


$f : \mathbb{R}^n \to \mathbb{R}^m$

$f(x, y, z) = (x + y, 2z)$

$$f\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Vector Transformation



$(1, 2, 3) \in \mathbb{R}^3 \Rightarrow (3, 6) \in \mathbb{R}^2$

$$\boxed{T : \mathbb{R}^3 \to \mathbb{R}^2}$$

Eg:

Defn

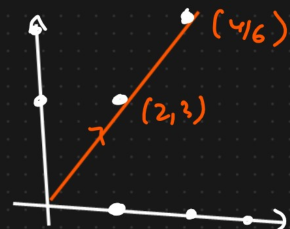
Vector transformations refer to operations that map vectors from one space to another, often changing their magnitude, direction, or both. These transformations are typically described using matrices and are fundamental in various fields, including physics, engineering, computer graphics, and data science.

Eg:

1) Scaling

Scaling is a transformation that changes the magnitude of vector while keeping their direction same.

$$v' = 2v = 2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$$



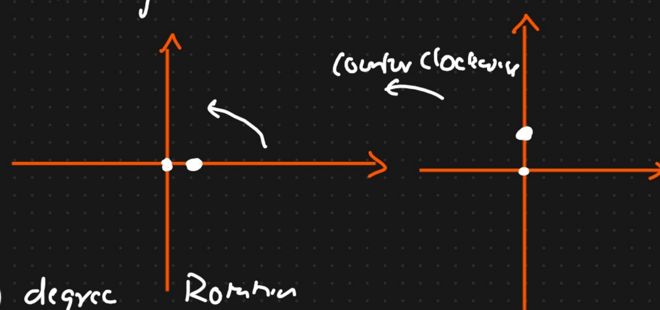
Application

1. DATA Normalization
2. Computer graphic to resize objects $\Rightarrow$ Paint $\Rightarrow$ Image $\Rightarrow$ Resize

② Rotation

Transformation that turns vectors around the origin.

$$v = \begin{bmatrix}1\\0\end{bmatrix} \in R^2$$



$$v' = \begin{bmatrix}0\\1\end{bmatrix}$$ $\Rightarrow$ Showing a 90 degree Rotation

Eg: Rotation will be used in Image processing $\Rightarrow$ Rotating Image.

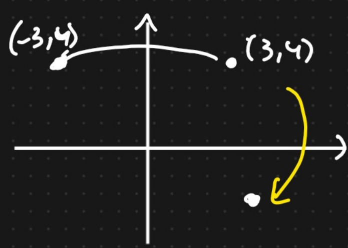
Robotics $\Rightarrow$ Adjusting Robot orientation

3D graphics $\Rightarrow$ Rotating objects.

③ Reflection

Transformation that flips vectors over a specified axis or plane.

$$v = \begin{bmatrix}3\\4\end{bmatrix}$$ $\Rightarrow$ Across the y axis. $\nearrow f(x)$.



$$\begin{bmatrix}3\\4\end{bmatrix} \cdot \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-3\\4\end{bmatrix}$$

④ Mirroring Images

Analyzing wave reflections.

④ Shearing

⊛ Linear Transformation

A linear transformation is a function between two vector spaces that preserves the operations of vector addition and scalar multiplication. This means that if T is a linear transformation from a vector space V to a vector space *W*, then for any vectors

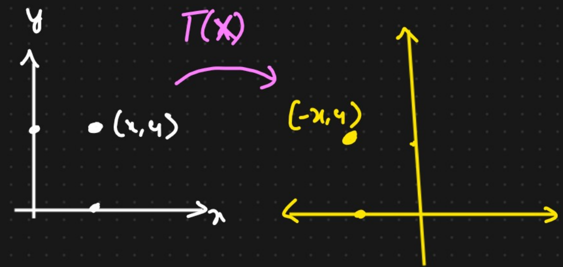
2 important properties

$T: V \to W \} \Rightarrow$ Linear Transformation

① Additivity $T(u+v) = T(u) + T(v)$

② Homogeneity $T(cu) = cT(u)$

$u, v \in V$ and c is a scalar value



Eg: Reflection

The reflection transformation T across the y axis maps a vector

$X = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ $\quad T(x) = \begin{bmatrix} -x \\ y \end{bmatrix}$

Transformation can be expressed as $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\rightarrow \boxed{T(x) = AX} \Rightarrow$ Linear Transformation

$$= \underset{1\times 2}{\begin{bmatrix} x & y \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{1\times 2}{\begin{bmatrix} -x \\ y \end{bmatrix}}$$

① Checking Additivity

Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vector in R^2

$$T(u+v) = T(u) + T(v)\}$$

$$u+v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$$

$$T(u+v) = A(u+v) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} -(u_1+v_1) \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} -u_1-v_2 \\ u_2+v_2 \end{bmatrix}$$

$$T(u) = Au = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix}$$

$$\boxed{LHS = RHS}$$

$$T(v) = Av = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$$

$$RHS \Rightarrow T(u) + T(v) = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -u_1-v_1 \\ u_2+v_2 \end{bmatrix}$$

2) Checking Homogenity

Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in R^2$ and c be a scaler

Homogenity Requirement

$$T(cu) = c\,T(u)$$

$$cu = c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$$

$$T(cu) = A(cu) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = \begin{bmatrix} -cu_1 \\ cu_2 \end{bmatrix} \Rightarrow LHS$$

Equal

RHS $\Uparrow$

$$c\,T(u) = c(A\cdot u) = c\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow c\begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -cu_1 \\ cu_2 \end{bmatrix}$$

Example that dont follow Linear Transformation

$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad T(x) = x + b \quad$ $T: R^2 \to R^2$

$u \quad v$

vector ⇒ fixed vector

$$T(x) = x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

① Check Additivity

$$T(u+v) = T(u) + T(v)$$

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad v = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$T(u+v) = T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix}\right)$$

$$T\Bigg(\underbrace{\begin{bmatrix} 6 \\ 2 \end{bmatrix}}_{x}\Bigg) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \Rightarrow LHS$$

$T(u) + T(v)$

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$T(u) + T(v) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \Rightarrow RHS$$

$LHS \neq RHS$

Check Homogeneity:

$$T(cu) = c\,T(u)$$

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad c = 2$$

$$T(cu) = T\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \Rightarrow \text{LHS}$$

$$\neq$$

$$c\,T(u) = 2\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \Rightarrow \text{RHS}$$

$$T(x) = x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{Not a Linear Transformation}$$

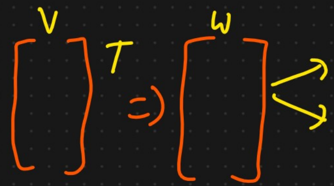
Fails both additivity and homogenity properties.

Why Linear Transformation?

Why Linear Transformation

Linear transformations are fundamental in data science for several reasons. They provide a mathematical framework for manipulating and analyzing data, which is crucial for various data processing tasks, model building, and interpretation. Here are some key reasons why linear transformations are important in data science:

1. Dimensionality Reduction
Principal Component Analysis (PCA): ⇒ 5000 ⇒ Lower Dimension ⇒ Variance
PCA is a widely used technique for reducing the dimensionality of datasets while retaining as much variance as possible. It involves finding a set of orthogonal axes (principal components) and projecting the data onto these axes. The transformation of data points in the original space to the new space defined by the principal components is a linear transformation. This helps in:



Reducing computational cost.
Mitigating the curse of dimensionality.
Visualizing high-dimensional data.

2. **Feature Engineering**
Linear transformations can be used to create new features from existing ones. For example, interactions between features can be captured through linear combinations, which can then be used in machine learning models to improve predictive performance. Techniques like linear regression, ridge regression, and linear discriminant analysis (LDA) all rely on linear transformations to find meaningful feature representations.

3. **Data Preprocessing**
Normalization and Standardization:
Linear transformations are used to scale data, making it suitable for machine learning models. Standardization transforms data to have a mean of zero and a standard deviation of one, while normalization scales data to a specific range (e.g., [0, 1]). These transformations are essential for ensuring that all features contribute equally to the model, especially in algorithms like gradient descent.

4. **Neural Networks** {Forward, Activation}
In neural networks, especially deep learning models, the layers consist of linear transformations followed by non-linear activation functions. The weights in a neural network can be seen as a series of linear transformations that map input data to intermediate layers and, eventually, to the output layer. This linear aspect is crucial for the network's ability to learn complex patterns in data.

5. **Image and Signal Processing**
In image and signal processing, linear transformations are used extensively. For example:

Convolutional filters in image processing can be seen as linear transformations applied to local regions of an image.
Fourier transforms, which decompose signals into sinusoidal components, are linear transformations that convert time-domain signals into frequency-domain representations.
6. Understanding and Interpretation
Linear transformations simplify complex relationships between variables into linear relationships, which are easier to understand and interpret. For example, linear regression provides a clear model of how each feature affects the target variable through linear coefficients, making it easier to explain to stakeholders.

7. **Optimization and Solving Systems of Equations**
Linear transformations are used to solve systems of linear equations, which is a common problem in data analysis and optimization. Techniques like matrix inversion and the use of pseudo-inverses are essential for finding solutions in linear regression and other linear models.

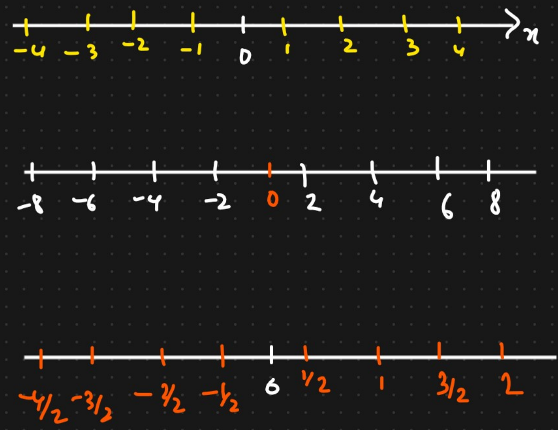
8. Theoretical Foundations
Many advanced machine learning algorithms and statistical techniques have linear algebra and linear transformations at their core. Understanding these fundamentals is crucial for grasping more complex topics like support vector machines

Linear Transformations Visualization

$T: R \to R$ 1 dimension

2d Matrix

$T(x) = 2x$ [Linear Transformation]



$T(x) = \frac{1}{2}x$

$f(x) = 2x$

$T(x) = -3x$

Property

1. Origin must be fixed
2. All lines must remain lines